

# NEW INTEGRABLE NONLOCAL NONLINEAR SCHRÖDINGER SYSTEMS FROM GEOMETRIC CURVE FLOWS IN $SO(2N)/U(N)$

AHMED M. G. AHMED AND STEPHEN C. ANCO

DEPARTMENT OF MATHEMATICS AND STATISTICS  
BROCK UNIVERSITY  
ST. CATHARINES, ON CANADA

**ABSTRACT.** A class of integrable nonlocal nonlinear Schrodinger systems with a unitary invariant bi-Hamiltonian formulation is derived by applying a general moving frame method to non-stretching curve flows in the symmetric space  $SO(2n)/U(n)$ . The systems involve a real scalar variable coupled to a pair of complex vectors variables which arise as Hasimoto variables defined by a parallel frame along the curves, where the equivalence group of the frame contains the factor  $U(1) \times SU(2)$ . The curve flow equations corresponding to these systems are shown to be non-stretching variants of Schrodinger maps. The same method can be applied to other symmetric spaces in which curves admit a parallel frame with a similar unitary equivalence group.

## 1. INTRODUCTION AND SUMMARY

Several integrable nonlocal generalizations of the nonlinear Schrodinger (NLS) equation have been obtained in recent work [1, 2]. These generalizations possess a Lax pair, a bi-Hamiltonian formulation, and a hierarchy of symmetries and conservation laws, all of which are  $U(1)$ -invariant. Each of the nonlocal NLS equations, together with its integrability structure, arises in a natural way from applying a geometrical moving frame method to flows of arclength-parameterized curves in certain low-dimensional geometric spaces, analogously to the well-known geometrical derivation of the NLS equation [3, 4, 5, 6] from flows of arclength-parameterized curves in Euclidean space,  $\mathbb{R}^3$ .

A key to the derivation of the NLS equation is the introduction of a parallel frame [7] which is geometrically determined by a curve up to the action of rigid  $SO(2)$  rotations in the normal plane of the curve. The components of the Cartan connection matrix of such a frame yield a complex-valued differential covariant of the curve, related to the pair of curvature and torsion invariants by the famous Hasimoto transformation [3]. For a curve undergoing a bi-normal flow in  $\mathbb{R}^3$ , the induced flow on this differential covariant turns out to be given by the NLS equation. Moreover, the Cartan structure equations of the frame are found to encode the NLS Lax pair as well as the NLS bi-Hamiltonian operators, where the  $U(1)$ -invariance of this integrability structure directly corresponds to the  $SO(2)$  equivalence group for parallel frames in  $\mathbb{R}^3$ .

The geometric spaces used in the deriving the nonlocal NLS equations are Euclidean symmetric spaces  $\mathbb{R}^m = (H \ltimes \mathbb{R}^m)/H$ . These spaces are flat Riemannian geometries in which a semisimple Lie group  $H$  is the gauge group of the frame bundle, and the semi-direct product group  $H \ltimes \mathbb{R}^m$  is the isometry group. Euclidean symmetric spaces have a natural construction starting from symmetric Lie algebras  $\mathfrak{g}/\mathfrak{h}$ , where  $\mathfrak{h}$  is the Lie algebra of  $H$  and

$\mathfrak{g} \supset \mathfrak{h}$  is the Lie algebra of a semisimple Lie group  $G \supset H$  that possesses an involutive automorphism under which  $H$  is invariant. The construction consists simply of defining the group action of  $H$  on  $\mathbb{R}^m$  to be the adjoint representation of  $H$  acting on the vector space  $\mathfrak{g}/\mathfrak{h}$ , with  $m = \dim \mathfrak{g}/\mathfrak{h}$ . We will use the notation  $Euc(\mathfrak{g}/\mathfrak{h}) = \mathbb{R}^m$  to denote these spaces.

One of the nonlocal NLS equations [1] is obtained from  $\mathbb{R}^5 = Euc(\mathfrak{su}(3)/\mathfrak{so}(3))$  which is a flat Riemannian geometry whose frame bundle has the gauge group  $SO(3)$ . The only other flat Riemannian geometry having the same gauge group is ordinary Euclidean space,  $\mathbb{R}^3 = Euc(\mathfrak{so}(4)/\mathfrak{so}(3))$ . As a consequence, underlying the derivation of the nonlocal NLS equation in  $\mathbb{R}^5 = Euc(\mathfrak{su}(3)/\mathfrak{so}(3))$  is the introduction of a naturally defined parallel frame with the same equivalence group  $SO(2) \simeq U(1) \subset SO(3)$  for arclength-parameterized curves as in  $\mathbb{R}^3 = Euc(\mathfrak{so}(4)/\mathfrak{so}(3))$ . Consequently, the nonlocal NLS equation arising from this space is a  $U(1)$ -invariant integrable system for a complex scalar variable.

The other recent generalizations of the NLS equation [2] arise from the spaces  $\mathbb{R}^4 = Euc(\mathfrak{su}(3)/\mathfrak{u}(2)) \simeq Euc(\mathfrak{su}(3)/\mathfrak{so}(3) \oplus \mathfrak{so}(2))$  and  $\mathbb{R}^6 = Euc(\mathfrak{sp}(2)/\mathfrak{u}(2)) \simeq Euc(\mathfrak{so}(5)/\mathfrak{so}(3) \oplus \mathfrak{so}(2))$ . These are flat Hermitian geometries sharing the gauge group  $SO(3) \times SO(2)$  for their respective frame bundles. For arclength-parameterized curves in both geometries, there is again a natural parallel frame whose equivalence group is  $SO(2) \simeq U(1) \subset SO(3) \times SO(2)$ , but compared to the previous geometries the components of the Cartan connection matrix of this frame yield a real-valued differential covariant of the curve in addition to a complex-valued differential covariant. The resulting generalizations of the NLS equation are  $U(1)$ -invariant integrable systems in which a real scalar variable is coupled to a complex scalar variable.

In the present paper we derive new multi-component nonlocal NLS equations from flows of arclength-parameterized curves in the Euclidean symmetric space  $\mathbb{R}^{n(n-1)} = Euc(\mathfrak{so}(2n)/\mathfrak{u}(n))$  for  $n \geq 3$ . This space is a flat Hermitian geometry having the unitary group  $U(n)$  as the gauge group of the frame bundle. (For  $n = 2$ , the space  $Euc(\mathfrak{so}(4)/\mathfrak{u}(2))$  is isometric to  $\mathbb{R}^2$ , so we do not consider this case further.)

Our derivation employs a parallel frame provided by the results in Ref. [8] for general Riemannian symmetric spaces  $M = G/H$ . In particular, Ref. [8] shows that the Cartan structure equations of suitably defined parallel frames explicitly encode a pair of compatible Hamiltonian operators with respect to Hasimoto variables that are given by the components of the Cartan connection matrix of this frame along curves in  $M = G/H$ . The Hasimoto variables have a geometrical meaning as differential covariants determined from a curve up to the action of the equivalence group of the frame. By applying these general results to curves in  $M = SO(2n)/U(n)$ , we obtain a natural parallel frame having the equivalence group  $SU(2) \times U(n-2) \subset U(n)$ , which contains a  $U(1)$  subgroup (the center of  $U(n-2)$ ) and a  $SU(2)$  subgroup. The corresponding Hasimoto variables consist of a real scalar variable and a pair of complex  $(n-2)$ -component vector variables. We also obtain an explicit pair of compatible Hamiltonian operators having  $SU(2) \times U(n-2)$  invariance.

Our main result is that we use the generators of the  $U(1)$  and  $SU(2)$  subgroups to derive a class of four different nonlocal integrable NLS systems for the Hasimoto variables. Each system displays  $U(n-2)$  invariance and has a bi-Hamiltonian formulation and a Lax pair, while three of the systems are related by an  $SU(2)$  transformation group. In this derivation the curvature of the space  $M = SO(2n)/U(n)$  is found to contribute only an inessential phase term in the NLS systems, and so all of our results carry over directly to the flat space

$\mathbb{R}^{n(n-1)} = \text{Euc}(\mathfrak{so}(2n)/\mathfrak{u}(n))$ . Moreover, the derivation uses only the Riemannian structure of these spaces (and not their Hermitian structure).

We show as a further result that the curve flows corresponding to the new nonlocal NLS systems are variants of Schrodinger map equations that exhibit geometrical invariance under the isometry group  $SO(2n)$  of the space  $M = SO(2n)/U(n)$  and that preserve the  $SO(2n)$ -invariant arclength locally along the curve (i.e. the flow is non-stretching). These geometrical curve flow equations in  $M = SO(2n)/U(n)$ , or in its flat counterpart  $\mathbb{R}^{n(n-1)} = \text{Euc}(\mathfrak{so}(2n)/\mathfrak{u}(n))$ , can be viewed as analogs of the vortex filament equation in  $\mathbb{R}^3$ .

In Sec. 2, we first review the construction and properties of moving parallel frames, Hasimoto variables, and Hamiltonian operators in Riemannian symmetric spaces, and then we apply these constructions to the space  $M = SO(2n)/U(n)$ . In Sec. 3, we show how to use the  $U(1)$  and  $SU(2)$  subgroups of the  $SU(2) \times U(n-2)$  invariance group of the Hamiltonian operators to derive the new nonlocal Schrodinger systems and their bi-Hamiltonian integrability structure. In Sec. 4, we work out the explicit curve flow equations corresponding to these integrable systems. We make some concluding remarks in Sec. 5. An appendix contains the essential algebraic properties of the Lie algebras  $\mathfrak{so}(2n) \supset \mathfrak{u}(n)$  which are used throughout the paper.

## 2. HAMILTONIAN OPERATORS

The symmetric space  $M = SO(2n)/U(n)$  has a natural Riemannian structure which comes from a soldering identification between the tangent space  $T_x M$  at points  $x$  and the vector space  $\mathfrak{m} = \mathfrak{so}(2n)/\mathfrak{u}(n)$ . This soldering  $T_x M \simeq \mathfrak{m}$  relies on the algebraic properties of  $\mathfrak{g} = \mathfrak{so}(2n) \supset \mathfrak{h} = \mathfrak{u}(n)$  as a symmetric Lie algebra. In particular,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{so}(2n), \quad \langle \mathfrak{h}, \mathfrak{m} \rangle = 0 \quad (1)$$

is an orthogonal direct sum decomposition relative to the Cartan-Killing form on  $\mathfrak{g} = \mathfrak{so}(2n)$ , with the Lie bracket relations

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} \quad (2)$$

induced from the Lie bracket on  $\mathfrak{g} = \mathfrak{so}(2n)$ . Note  $\dim \mathfrak{m} = n(n-1)$  and  $\dim \mathfrak{h} = n^2$ .

A simple formulation of the soldering identification is provided by [9] a  $\mathfrak{m}$ -valued linear coframe  $e$  and a  $\mathfrak{h}$ -valued linear connection  $\omega$  whose torsion and curvature

$$\mathfrak{T} := de + [\omega, e], \quad \mathfrak{R} := d\omega + \frac{1}{2}[\omega, \omega] \quad (3)$$

are 2-forms with respective values in  $\mathfrak{m}$  and  $\mathfrak{h}$ , given by the Cartan structure equations

$$\mathfrak{T} = 0, \quad \mathfrak{R} = -\frac{1}{2}[e, e]. \quad (4)$$

Here  $[\cdot, \cdot]$  denotes the Lie bracket on  $\mathfrak{g}$  composed with the wedge product on  $T_x^* M$ . This structure together with the (negative-definite) Cartan-Killing form determines a Riemannian metric  $g$  and a Riemannian connection (i.e. covariant derivative)  $\nabla$  on the space  $M = SO(2n)/U(n)$  from the following soldering relations:

$$g(X, Y) := -\langle e_X, e_Y \rangle \quad (5)$$

$$e] \nabla_X Y := \partial_X e_Y + [\omega_X, e_Y] \quad (6)$$

for all  $X, Y$  in  $T_x M$ , where  $e \rfloor X = e_X, e \rfloor Y = e_Y \in \mathfrak{m}$ . In addition, the 2-forms  $\mathfrak{T}$  and  $\mathfrak{R}$  determine the torsion tensor  $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$  and the curvature tensor  $R(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$  through the soldering relations

$$e \rfloor T(X, Y) = \mathfrak{T} \rfloor (X \wedge Y) = 0 \quad (7)$$

$$e \rfloor R(X, Y)Z = [\mathfrak{R} \rfloor (X \wedge Y), e_Z] = -[[e_X, e_Y], e_Z]. \quad (8)$$

These expressions (5)–(8) show that  $\nabla$  is metric compatible,  $\nabla g = 0$ , and torsion-free,  $T = 0$ , while its curvature is covariantly constant,  $\nabla R = 0$ .

This formulation of the Riemannian structure of  $M = SO(2n)/U(n)$  has an intrinsic gauge freedom consisting of the transformations

$$e \longrightarrow \text{Ad}(f^{-1})e, \quad \omega \longrightarrow \text{Ad}(f^{-1})\omega + f^{-1}df \quad (9)$$

as defined in terms of an arbitrary function  $f : M \rightarrow H = U(n) \subset G = SO(2n)$ . The gauge transformations (9) comprise a local ( $x$ -dependent) representation of the linear transformation group  $H^* = \text{Ad}(U(n))$  which defines the gauge group [10] of the frame bundle of  $M$ . Both the metric tensor  $g$  and curvature tensor  $R$  on  $M$  are gauge invariant.

Geometrically, the group  $G = SO(2n)$  represents the isometry group of  $M$ , while the subgroup  $H = U(n)$  represents the isotropy subgroup of the origin  $o$  in  $M$ . In terms of the symmetric Lie algebra structure (2), the Lie subalgebra  $\mathfrak{h}$  is identified with the generators of isometries that leave fixed the origin  $o$  in  $M$ , and the vector space  $\mathfrak{m}$  is identified with the generators of isometries that carry the origin  $o$  to any point  $x \neq o$  in  $M$ .

We remark that  $M = SO(2n)/U(n)$  also has a Hermitian structure, consisting of a complex structure tensor  $J$  that satisfies the properties  $J^2 = -\text{id}$ ,  $\nabla J = 0$ , and  $g(JX, JY) = g(X, Y)$  for all  $X, Y$  in  $T_x M$ . However, this structure will not be used in what follows.

Let  $\gamma(x)$  be any smooth curve in  $M = SO(2n)/U(n)$ . A moving frame consists of a set of orthonormal vectors that span the tangent space  $T_\gamma M$  at each point  $x$  on the curve  $\gamma$ . The Frenet equations of a moving frame yield a connection matrix consisting of the set of frame components of the covariant  $x$ -derivative of each frame vector along the curve [11]. A moving coframe consists of a set of orthonormal covectors that are dual to the frame vectors relative to the Riemannian metric  $g$ . Such a framing for  $\gamma(x)$  is determined by the Lie-algebra components of  $e$  and  $\omega \rfloor \gamma_x$  when an orthonormal basis is introduced for  $\mathfrak{m} = \mathfrak{so}(2n)$  and  $\mathfrak{h} = \mathfrak{u}(n)$  with respect to the Cartan-Killing form, where the Frenet equations are defined by the frame components of the transport equation

$$\nabla_x e = -\text{ad}(\omega \rfloor \gamma_x)e \quad (10)$$

along the curve. In particular, if  $\{\mathbf{m}_l\}$ ,  $l = 1, \dots, n(n-1)$ , is any fixed orthonormal basis for  $\mathfrak{m}$ , then a frame at each point  $x$  along the curve is given by the set of vectors  $X_l := -\langle e^*, \mathbf{m}_l \rangle$ ,  $l = 1, \dots, n(n-1)$ . Here  $e^*$  is a  $\mathfrak{m}$ -valued linear frame defined as the dual to the linear coframe  $e$  by the condition that  $-\langle e^*, e \rangle = \text{id}$  is the identity map on each tangent space  $T_x M$  (cf [8, 9]).

Now consider any smooth flow  $\gamma(t, x)$  of a curve in  $M = SO(2n)/U(n)$ . We write  $X = \gamma_x$  for the tangent vector and  $Y = \gamma_t$  for the evolution vector at each point  $x$  along the curve. The flow is *non-stretching* provided that it preserves the  $SO(2n)$ -invariant arclength  $ds = |\gamma_x|dx$ , or equivalently

$$\nabla_t |\gamma_x| = 0 \quad (11)$$

in which case we have  $g(\gamma_x, \gamma_x) = |\gamma_x|^2 = 1$  without loss of generality. For flows that are transverse to the curve (such that  $X$  and  $Y$  are linearly independent),  $\gamma(t, x)$  will describe a smooth two-dimensional surface in  $M$ . The pullback of the torsion and curvature equations (4) to this surface yields

$$D_x h - D_t e + [u, h] - [\varpi, e] = 0, \quad (12)$$

$$D_x \varpi - D_t u + [u, \varpi] = -[e, h], \quad (13)$$

with

$$e := e \rfloor X = e \rfloor \gamma_x, \quad h := e \rfloor Y = e \rfloor \gamma_t, \quad (14)$$

$$u := \omega \rfloor X = \omega \rfloor \gamma_x, \quad \varpi := \omega \rfloor Y = \omega \rfloor \gamma_t, \quad (15)$$

where  $D_x, D_t$  denote derivative operators with respect to  $x, t$ . For any non-stretching curve flow, these structure equations (12)–(15) encode a pair of compatible Hamiltonian operators which can be written in an explicit form in terms of an  $U(n)$ -parallel framing for  $\gamma(t, x)$  as follows.

A  $U(n)$ -parallel frame along a curve in  $M = SO(2n)/U(n)$  is a direct algebraic generalization of a parallel moving frame in Euclidean geometry [7], as defined by the properties [8]:

- (i)  $e$  is a constant unit-norm element lying in a Cartan subspace  $\mathfrak{a} \subset \mathfrak{m} = \mathfrak{so}(2n)$  that is contained in the centralizer subspace  $\mathfrak{m}_\parallel$  of  $e$ , i.e.  $D_x e = D_t e = 0$ ,  $\langle e, e \rangle = -1$ , and  $\text{ad}(\mathfrak{m}_\parallel)e = 0$  where  $\mathfrak{m}_\parallel \oplus \mathfrak{m}_\perp = \mathfrak{m} = \mathfrak{so}(2n)$  and  $\langle \mathfrak{m}_\parallel, \mathfrak{m}_\perp \rangle = 0$ .
- (ii)  $u$  lies in the perp space  $\mathfrak{h}_\perp$  of the Lie subalgebra  $\mathfrak{h}_\parallel \subset \mathfrak{h} = \mathfrak{u}(n)$  of the linear isotropy group  $H_\parallel^* \subset H^* = \text{Ad}(U(n))$  that preserves  $e$ , i.e.  $\text{ad}(\mathfrak{h}_\parallel)e = 0$  and  $\langle u, \mathfrak{h}_\parallel \rangle = 0$  where  $\mathfrak{h}_\parallel \oplus \mathfrak{h}_\perp = \mathfrak{h} = \mathfrak{u}(n)$  and  $\langle \mathfrak{h}_\parallel, \mathfrak{h}_\perp \rangle = 0$ .

Existence of such moving frames can be established by applying a suitable gauge transformation (9) to an arbitrary linear frame at each point  $x$  along the curve [8]. The necessary transformation is unique up to a rigid gauge freedom given by  $x$ -independent transformations in  $H_\parallel^* \subset H^* = \text{Ad}(U(n))$  preserving the tangent vector  $X$ . This rigid gauge freedom is called the *equivalence group* of the frame.

The properties and construction of  $U(n)$ -parallel frames rely on the Lie bracket relations for the subspaces  $\mathfrak{m}_\parallel, \mathfrak{m}_\perp, \mathfrak{h}_\parallel, \mathfrak{h}_\perp$  coming from the structure of  $\mathfrak{g}$  as a symmetric Lie algebra (2). These relations consist of

$$[\mathfrak{m}_\parallel, \mathfrak{m}_\parallel] \subseteq \mathfrak{h}_\parallel, \quad [\mathfrak{m}_\parallel, \mathfrak{h}_\parallel] \subseteq \mathfrak{m}_\parallel, \quad [\mathfrak{h}_\parallel, \mathfrak{h}_\parallel] \subseteq \mathfrak{h}_\parallel, \quad (16)$$

$$[\mathfrak{h}_\parallel, \mathfrak{m}_\perp] \subseteq \mathfrak{m}_\perp, \quad [\mathfrak{h}_\parallel, \mathfrak{h}_\perp] \subseteq \mathfrak{h}_\perp, \quad (17)$$

$$[\mathfrak{m}_\parallel, \mathfrak{m}_\perp] \subseteq \mathfrak{h}_\perp, \quad [\mathfrak{m}_\parallel, \mathfrak{h}_\perp] \subseteq \mathfrak{m}_\perp, \quad (18)$$

while the remaining Lie brackets obey the general relations

$$[\mathfrak{m}_\perp, \mathfrak{m}_\perp] \subset \mathfrak{h}, \quad [\mathfrak{h}_\perp, \mathfrak{h}_\perp] \subset \mathfrak{h}, \quad [\mathfrak{h}_\perp, \mathfrak{m}_\perp] \subset \mathfrak{m}. \quad (19)$$

Through property (i), the set of smooth curves  $\gamma(x)$  in  $M = SO(2n)/U(n)$  can be divided into algebraic equivalence classes defined by the orbit of the element  $e \rfloor \gamma_x = e$  in  $\mathfrak{a} \subset \mathfrak{m}$  under the action of the group  $H^* = \text{Ad}(H)$ .

We will consider a non-stretching curve flow  $\gamma(t, x)$  in  $M = SO(2n)/U(n)$  having a  $U(n)$ -parallel framing in which the tangential components (14) of the linear coframe and the linear

connection are given by the variables

$$e = \frac{1}{\sqrt{\chi}}(1, (\mathbf{0}, \mathbf{0})) \in \mathfrak{a} \subset \mathfrak{m}_{\parallel}, \quad \chi = 8(n-1) \quad (20)$$

$$u = (u, (\mathbf{u}_{1,1}, \mathbf{u}_{1,2}, \mathbf{u}_{2,1}, \mathbf{u}_{2,2})) \in \mathfrak{h}_{\perp} \quad (21)$$

while the flow components (14) of the linear coframe and the linear connection are expressed in terms of the variables

$$h_{\parallel} = (h_{\parallel}, (\mathbf{H}_{1\parallel}, \mathbf{H}_{2\parallel})) \in \mathfrak{m}_{\parallel} \quad (22)$$

$$h_{\perp} = (h_{\perp}, (\mathbf{h}_{1,1}, \mathbf{h}_{1,2}, \mathbf{h}_{2,1}, \mathbf{h}_{2,2})) \in \mathfrak{m}_{\perp} \quad (23)$$

$$\varpi^{\parallel} = ((\theta_1, \theta_2, \theta), (\Theta_1, \Theta_2)) \in \mathfrak{h}_{\parallel} \quad (24)$$

$$\varpi^{\perp} = (w, (\mathbf{w}_{1,1}, \mathbf{w}_{1,2}, \mathbf{w}_{2,1}, \mathbf{w}_{2,2})) \in \mathfrak{h}_{\perp} \quad (25)$$

using the matrix identifications (217)–(224) shown in the appendix. Here  $u, h_{\parallel}, h, \theta_1, \theta_2, \theta, w \in \mathbb{R}$  are real scalar variables,  $\mathbf{w}_{1,1}, \mathbf{w}_{1,2}, \mathbf{w}_{2,1}, \mathbf{w}_{2,2}, \mathbf{h}_{1,1}, \mathbf{h}_{1,2}, \mathbf{h}_{2,1}, \mathbf{h}_{2,2} \in \mathbb{R}^{n-2}$  are real vector variables,  $\mathbf{H}_{1\parallel}, \mathbf{H}_{2\parallel}, \Theta_1 \in \mathfrak{so}(n-2)$  are anti-symmetric matrix variables,  $\Theta_2 \in \mathfrak{s}(n-2)$  is a real symmetric matrix variable. For later use, we also introduce the variable

$$\begin{aligned} h^{\perp} &= (h^{\perp}, (\mathbf{h}^{1,1}, \mathbf{h}^{1,2}, \mathbf{h}^{2,1}, \mathbf{h}^{2,2})) = \text{ad}(e)h_{\perp} \in \mathfrak{h}_{\perp} \\ &= \frac{1}{\sqrt{\chi}}(-2h_{\perp}, (\mathbf{h}_{2,1}, \mathbf{h}_{2,2}, -\mathbf{h}_{1,1}, -\mathbf{h}_{1,2})) \end{aligned} \quad (26)$$

where  $\mathbf{h}^{1,1}, \mathbf{h}^{1,2}, \mathbf{h}^{2,1}, \mathbf{h}^{2,2} \in \mathbb{R}^{n-2}$  are real vector variables.

Since  $T_{\gamma}M$  has rank  $\lfloor n/2 \rfloor \geq 1$ , if  $n = 2$  then all non-stretching curve flows  $\gamma(t, x)$  in  $M = SO(2n)/U(n)$  belong to the same algebraic equivalence class, corresponding to the element (20), while if  $n > 2$  then the element (20) determines one particular algebraic equivalence class of non-stretching curve flows in  $M = SO(2n)/U(n)$ .

As shown in the appendix, the  $U(n)$ -parallel frame (20)–(21) has the equivalence group

$$H_{\parallel}^* = \text{Ad}(SU(2) \times U(n-2)) \subset \text{Ad}(U(n)) \quad (27)$$

whose action consists of rigid ( $x$ -independent) gauge transformations (9) given by the matrix representation (240)–(243). The resulting coframe  $e$  provides an isomorphism between  $T_{\gamma}M$  and  $\mathfrak{m} \simeq \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}^{n-2} \oplus \mathbb{R}^{n-2} \oplus \mathbb{R}^{n-2} \oplus \mathbb{R}^{n-2} \oplus \mathfrak{so}(n-2) \oplus \mathfrak{so}(n-2)$ , which yields a correspondence between the vectors  $\{X_l\}$  in a moving frame for  $T_{\gamma}M$  and the vectors  $\{\mathbf{m}_l\}$  in a basis for  $\mathfrak{m}$ . In particular, if  $e$  is chosen to be one of the basis vectors, then the tangent vector  $X = \gamma_x$  of the curve is one of the vectors in the moving frame.

Because this framing has a non-trivial equivalence group, the components (21) of the linear connection  $\omega|_{\gamma_x}$  geometrically define a *covariant* of the curve  $\gamma$  relative to the equivalence group (27). Likewise,  $x$ -derivatives of these components geometrically describe *differential covariants* of  $\gamma$  [8]. We note that the geometric invariants of  $\gamma$  as defined by Riemannian inner products of the tangent vector  $X = \gamma_x$  with the principal normal vector  $N = \nabla_x \gamma_x$  and its covariant  $x$ -derivatives  $\nabla_x N = \nabla_x^2 \gamma_x, \nabla_x^3 N = \nabla_x^3 \gamma_x, \dots$  can be expressed as scalars formed from Cartan-Killing inner products of the covariant  $u$  and differential covariants  $u_x, u_{xx}, \dots$ . In particular, the set of invariants given by  $\{g(X, \nabla_x^l X)\}$ ,  $l = 1, \dots, n^2 - n - 1 (= \dim \mathfrak{m} - 1)$ , generates the components of the connection matrix of a classical Frenet frame [11] determined by the curve  $\gamma$ .

**2.1. Non-stretching curve flow equations in a  $U(n)$ -parallel frame.** By projecting the Cartan structure equations (12)–(13) into the subspaces  $\mathfrak{m}_\perp$ ,  $\mathfrak{m}_\parallel$ ,  $\mathfrak{h}_\perp$ ,  $\mathfrak{h}_\parallel$ , we obtain the system

$$0 = D_x h_\perp + [u, h_\parallel] + [u, h_\perp]_\parallel - [\varpi^\perp, e] \in \mathfrak{m}_\perp, \quad (28)$$

$$0 = D_x h_\parallel + [u, h_\perp]_\parallel \in \mathfrak{m}_\parallel, \quad (29)$$

$$0 = D_x \varpi^\perp - D_t u + [u, \varpi^\parallel] + [u, \varpi^\perp]_\perp + [e, h_\perp] \in \mathfrak{h}_\perp, \quad (30)$$

$$0 = D_x \varpi^\parallel + [u, \varpi^\perp]_\parallel \in \mathfrak{h}_\parallel, \quad (31)$$

where

$$h = h_\parallel + h_\perp \in \mathfrak{m}_\parallel \oplus \mathfrak{m}_\perp, \quad \varpi = \varpi^\parallel + \varpi^\perp \in \mathfrak{h}_\parallel \oplus \mathfrak{h}_\perp. \quad (32)$$

This system can be written out explicitly as matrix component equations in terms of the variables (20)–(26):

$$\begin{aligned} w &= \frac{1}{2}\chi(\frac{1}{2}D_x h^\perp + \mathbf{u}_{1,1} \cdot \mathbf{h}^{1,2} + \mathbf{u}_{2,1} \cdot \mathbf{h}^{2,2} - \mathbf{u}_{1,2} \cdot \mathbf{h}^{1,1} - \mathbf{u}_{2,2} \cdot \mathbf{h}^{2,1}) + \sqrt{\chi} u h_\parallel \\ \mathbf{w}_{1,1} &= \chi(D_x \mathbf{h}^{1,1} + u \mathbf{h}^{1,2} + \frac{1}{2} h^\perp \mathbf{u}_{1,2}) + \sqrt{\chi}(h_\parallel \mathbf{u}_{1,1} + \mathbf{u}_{2,2} \lrcorner \mathbf{H}_{2\parallel} + \mathbf{u}_{2,1} \lrcorner \mathbf{H}_{1\parallel}) \\ \mathbf{w}_{1,2} &= \chi(D_x \mathbf{h}^{1,2} - u \mathbf{h}^{1,1} - \frac{1}{2} h^\perp \mathbf{u}_{1,1}) + \sqrt{\chi}(h_\parallel \mathbf{u}_{1,2} + \mathbf{u}_{2,1} \lrcorner \mathbf{H}_{2\parallel} - \mathbf{u}_{2,2} \lrcorner \mathbf{H}_{1\parallel}) \\ \mathbf{w}_{2,1} &= \chi(D_x \mathbf{h}^{2,1} + u \mathbf{h}^{2,2} + \frac{1}{2} h^\perp \mathbf{u}_{2,2}) + \sqrt{\chi}(h_\parallel \mathbf{u}_{2,1} - \mathbf{u}_{1,1} \lrcorner \mathbf{H}_{1\parallel} - \mathbf{u}_{1,2} \lrcorner \mathbf{H}_{2\parallel}) \\ \mathbf{w}_{2,2} &= \chi(D_x \mathbf{h}^{2,2} - u \mathbf{h}^{2,1} - \frac{1}{2} h^\perp \mathbf{u}_{2,1}) + \sqrt{\chi}(h_\parallel \mathbf{u}_{2,2} + \mathbf{u}_{1,2} \lrcorner \mathbf{H}_{1\parallel} - \mathbf{u}_{1,1} \lrcorner \mathbf{H}_{2\parallel}) \end{aligned} \quad (33)$$

$$\begin{aligned} D_x h_\parallel &= \sqrt{\chi}(u h^\perp + \mathbf{u}_{1,1} \cdot \mathbf{h}^{1,1} + \mathbf{u}_{2,1} \cdot \mathbf{h}^{2,1} + \mathbf{u}_{1,2} \cdot \mathbf{h}^{1,2} + \mathbf{u}_{2,2} \cdot \mathbf{h}^{2,2}) \\ D_x \mathbf{H}_{1\parallel} &= \sqrt{\chi}(-\mathbf{u}_{1,1} \wedge \mathbf{h}^{2,1} + \mathbf{u}_{2,1} \wedge \mathbf{h}^{1,1} + \mathbf{u}_{1,2} \wedge \mathbf{h}^{2,2} - \mathbf{u}_{2,2} \wedge \mathbf{h}^{1,2}) \\ D_x \mathbf{H}_{2\parallel} &= \sqrt{\chi}(-\mathbf{u}_{1,1} \wedge \mathbf{h}^{2,2} + \mathbf{u}_{2,1} \wedge \mathbf{h}^{1,2} - \mathbf{u}_{1,2} \wedge \mathbf{h}^{2,1} + \mathbf{u}_{2,2} \wedge \mathbf{h}^{1,1}) \end{aligned} \quad (34)$$

$$\begin{aligned} D_t u &= h^\perp + D_x w + \mathbf{u}_{1,1} \cdot \mathbf{w}_{1,2} - \mathbf{u}_{1,2} \cdot \mathbf{w}_{1,1} + \mathbf{u}_{2,1} \cdot \mathbf{w}_{2,2} - \mathbf{u}_{2,2} \cdot \mathbf{w}_{2,1} \\ D_t \mathbf{u}_{1,1} &= \mathbf{h}^{1,1} + D_x \mathbf{w}_{1,1} + \mathbf{u}_{1,1} \lrcorner \Theta_1 + (w + \theta) \mathbf{u}_{1,2} - \theta_1 \mathbf{u}_{2,1} + \theta_2 \mathbf{u}_{2,2} - u \mathbf{w}_{1,2} - \mathbf{u}_{1,2} \lrcorner \Theta_2 \\ D_t \mathbf{u}_{1,2} &= \mathbf{h}^{1,2} + D_x \mathbf{w}_{1,2} + \mathbf{u}_{1,1} \lrcorner \Theta_2 - (w + \theta) \mathbf{u}_{1,1} - \theta_2 \mathbf{u}_{2,1} - \theta_1 \mathbf{u}_{2,2} + u \mathbf{w}_{1,1} + \mathbf{u}_{1,2} \lrcorner \Theta_1 \end{aligned} \quad (35)$$

$$\begin{aligned} D_t \mathbf{u}_{2,1} &= \mathbf{h}^{2,1} + D_x \mathbf{w}_{2,1} + \mathbf{u}_{2,1} \lrcorner \Theta_1 + (w - \theta) \mathbf{u}_{2,2} + \theta_1 \mathbf{u}_{1,1} + \theta_2 \mathbf{u}_{1,2} - u \mathbf{w}_{2,2} - \mathbf{u}_{2,2} \lrcorner \Theta_2 \\ D_t \mathbf{u}_{2,2} &= \mathbf{h}^{2,2} + D_x \mathbf{w}_{2,2} + \mathbf{u}_{2,1} \lrcorner \Theta_2 - (w - \theta) \mathbf{u}_{2,1} - \theta_2 \mathbf{u}_{1,1} + \theta_1 \mathbf{u}_{1,2} + u \mathbf{w}_{2,1} + \mathbf{u}_{2,2} \lrcorner \Theta_1 \end{aligned}$$

$$\begin{aligned} D_x \theta_1 &= \mathbf{u}_{1,1} \cdot \mathbf{w}_{2,1} + \mathbf{u}_{1,2} \cdot \mathbf{w}_{2,2} - \mathbf{u}_{2,1} \cdot \mathbf{w}_{1,1} - \mathbf{u}_{2,2} \cdot \mathbf{w}_{1,2} \\ D_x \theta_2 &= -\mathbf{u}_{1,1} \cdot \mathbf{w}_{2,2} + \mathbf{u}_{1,2} \cdot \mathbf{w}_{2,1} - \mathbf{u}_{2,1} \cdot \mathbf{w}_{1,2} + \mathbf{u}_{2,2} \cdot \mathbf{w}_{1,1} \\ D_x \theta &= -\mathbf{u}_{1,1} \cdot \mathbf{w}_{1,2} + \mathbf{u}_{1,2} \cdot \mathbf{w}_{1,1} + \mathbf{u}_{2,1} \cdot \mathbf{w}_{2,2} - \mathbf{u}_{2,2} \cdot \mathbf{w}_{2,1} \\ D_x \Theta_1 &= \mathbf{u}_{1,1} \wedge \mathbf{w}_{1,1} + \mathbf{u}_{1,2} \wedge \mathbf{w}_{1,2} + \mathbf{u}_{2,1} \wedge \mathbf{w}_{2,1} + \mathbf{u}_{2,2} \wedge \mathbf{w}_{2,2} \\ D_x \Theta_2 &= \mathbf{u}_{1,1} \odot \mathbf{w}_{1,2} - \mathbf{u}_{1,2} \odot \mathbf{w}_{1,1} + \mathbf{u}_{2,1} \odot \mathbf{w}_{2,2} - \mathbf{u}_{2,2} \odot \mathbf{w}_{2,1} \end{aligned} \quad (36)$$

with the outer product notation

$$a \wedge b = a \otimes b - b \otimes a, \quad a \odot b = a \otimes b + b \otimes a. \quad (37)$$

Through equations (34) and (36), the variables  $h_{\parallel}$ ,  $\mathbf{H}_{1\parallel}$ ,  $\mathbf{H}_{2\parallel}$ ,  $\theta_1$ ,  $\theta_2$ ,  $\theta$ ,  $\Theta_1$ ,  $\Theta_2$  can be eliminated, yielding

$$\begin{pmatrix} u \\ \mathbf{u}_{1,1} \\ \mathbf{u}_{1,2} \\ \mathbf{u}_{2,1} \\ \mathbf{u}_{2,2} \end{pmatrix}_t = \mathcal{H} \begin{pmatrix} w \\ \mathbf{w}_{1,1} \\ \mathbf{w}_{1,2} \\ \mathbf{w}_{2,1} \\ \mathbf{w}_{2,2} \end{pmatrix} + \begin{pmatrix} h^{\perp} \\ \mathbf{h}^{1,1} \\ \mathbf{h}^{1,2} \\ \mathbf{h}^{2,1} \\ \mathbf{h}^{2,2} \end{pmatrix}, \quad \begin{pmatrix} w \\ \mathbf{w}_{1,1} \\ \mathbf{w}_{1,2} \\ \mathbf{w}_{2,1} \\ \mathbf{w}_{2,2} \end{pmatrix} = \mathcal{J} \begin{pmatrix} h^{\perp} \\ \mathbf{h}^{1,1} \\ \mathbf{h}^{1,2} \\ \mathbf{h}^{2,1} \\ \mathbf{h}^{2,2} \end{pmatrix} \quad (38)$$

in terms of some  $5 \times 5$  matrix operators  $\mathcal{H}$  and  $\mathcal{J}$ . These matrix equations (38) are invariant under the equivalence group  $H_{\parallel}^* = \text{Ad}(SU(2) \times U(n-2)) \subset \text{Ad}(U(n))$  of the  $U(n)$ -parallel frame. From general results in Ref. [8],  $\mathcal{H}$  is a cosymplectic Hamiltonian operator and  $\mathcal{J}$  is a symplectic Hamiltonian operator, such that  $\mathcal{H}$  and  $\mathcal{J}^{-1}$  are a bi-Hamiltonian pair of  $SU(2) \times U(n-2)$  invariant operators. We will reformulate this structure in terms of the following variables in  $\mathfrak{m}_{\perp}$ .

First we replace  $u$  and  $\varpi$  by the variables

$$\begin{aligned} v &= (v, (\mathbf{v}_{1,1}, \mathbf{v}_{1,2}, \mathbf{v}_{2,1}, \mathbf{v}_{2,2})) = (1/\sqrt{\chi})\text{ad}(e)^{-1}u \in \mathfrak{m}_{\perp} \\ &= (-\tfrac{1}{2}u, (-\mathbf{u}_{2,1}, -\mathbf{u}_{2,2}, \mathbf{u}_{1,1}, \mathbf{u}_{1,2})) \end{aligned} \quad (39)$$

$$\begin{aligned} \varpi_{\perp} &= (\omega, (\boldsymbol{\omega}_{1,1}, \boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{2,1}, \boldsymbol{\omega}_{2,2})) = -\sqrt{\chi}\text{ad}(e)\varpi^{\perp} \in \mathfrak{m}_{\perp} \\ &= (-2w, (-\mathbf{w}_{2,1}, -\mathbf{w}_{2,2}, \mathbf{w}_{1,1}, \mathbf{w}_{1,2})). \end{aligned} \quad (40)$$

Note these transformations on  $u$  and  $\varpi^{\perp}$  are inverses, so consequently the two operators  $\mathcal{H}$  and  $\mathcal{J}^{-1}$  will transform in the same way as each other, as seen from the pair of matrix equations (38) combined with the previous transformation (26) on  $h_{\perp}$ .

Next we combine the components of  $v, \varpi, h_{\perp}, h_{\parallel}, \varpi^{\parallel}$  so as to get complex variables

$$\mathbf{v}_1 = \mathbf{v}_{1,1} + i\mathbf{v}_{1,2}, \quad \mathbf{v}_2 = \mathbf{v}_{2,1} + i\mathbf{v}_{2,2} \quad (41)$$

$$\boldsymbol{\omega}_1 = \boldsymbol{\omega}_{1,1} + i\boldsymbol{\omega}_{1,2}, \quad \boldsymbol{\omega}_2 = \boldsymbol{\omega}_{2,1} + i\boldsymbol{\omega}_{2,2} \quad (42)$$

$$\mathbf{h}_1 = \mathbf{h}_{1,1} + i\mathbf{h}_{1,2}, \quad \mathbf{h}_2 = \mathbf{h}_{2,1} + i\mathbf{h}_{2,2} \quad (43)$$

$$\mathbf{H}_{\parallel} = \mathbf{H}_{1\parallel} + i\mathbf{H}_{2\parallel} \quad (44)$$

$$\Theta = \theta_1 + i\theta_2, \quad \Theta = \Theta_1 + i\Theta_2 \quad (45)$$

which will be seen to have a natural unitary transformation under the action of the equivalence group. In terms of these variables (41)–(45), the Cartan structure equations (33)–(36) become

$$\begin{aligned} D_t \mathbf{v} &= \tfrac{1}{4}D_x \omega + (1/\sqrt{\chi})h_{\perp} - \tfrac{1}{2}\text{Im}(\bar{\mathbf{v}}_2 \cdot \boldsymbol{\omega}_2 + \bar{\mathbf{v}}_1 \cdot \boldsymbol{\omega}_1) \\ D_t \mathbf{v}_1 &= D_x \boldsymbol{\omega}_1 + (1/\sqrt{\chi})\mathbf{h}_1 + i(\tfrac{1}{2}\omega + \theta)\mathbf{v}_1 - i2\mathbf{v}\boldsymbol{\omega}_1 + \mathbf{v}_1 \rfloor \Theta - \bar{\Theta}\mathbf{v}_2 \\ D_t \mathbf{v}_2 &= D_x \boldsymbol{\omega}_2 + (1/\sqrt{\chi})\mathbf{h}_2 + i(\tfrac{1}{2}\omega - \theta)\mathbf{v}_2 - i2\mathbf{v}\boldsymbol{\omega}_2 + \mathbf{v}_2 \rfloor \Theta + \Theta\mathbf{v}_1 \end{aligned} \quad (46)$$

$$\begin{aligned} \omega &= \sqrt{\chi}(D_x h_{\perp} + 4\mathbf{v}h_{\parallel} - \text{Im}(\bar{\mathbf{v}}_1 \cdot \mathbf{h}_1 + \bar{\mathbf{v}}_2 \cdot \mathbf{h}_2)) \\ \boldsymbol{\omega}_1 &= \sqrt{\chi}(D_x \mathbf{h}_1 + (h_{\parallel} + i h_{\perp})\mathbf{v}_1 + i2\mathbf{v}\mathbf{h}_1 + \bar{\mathbf{v}}_2 \rfloor \mathbf{H}_{\parallel}) \\ \boldsymbol{\omega}_2 &= \sqrt{\chi}(D_x \mathbf{h}_2 + (h_{\parallel} + i h_{\perp})\mathbf{v}_2 + i2\mathbf{v}\mathbf{h}_2 - \bar{\mathbf{v}}_1 \rfloor \mathbf{H}_{\parallel}) \end{aligned} \quad (47)$$



$$\begin{aligned}
D_x \Theta &= \bar{\mathbf{v}}_1 \cdot \boldsymbol{\omega}_2 - \bar{\boldsymbol{\omega}}_1 \cdot \mathbf{v}_2 \\
D_x \theta &= \text{Im}(\bar{\mathbf{v}}_1 \cdot \boldsymbol{\omega}_1 + \bar{\boldsymbol{\omega}}_2 \cdot \mathbf{v}_2) \\
D_x \boldsymbol{\Theta} &= \bar{\mathbf{v}}_2 \otimes \boldsymbol{\omega}_2 - \bar{\boldsymbol{\omega}}_2 \otimes \mathbf{v}_2 + \bar{\mathbf{v}}_1 \otimes \boldsymbol{\omega}_1 - \bar{\boldsymbol{\omega}}_1 \otimes \mathbf{v}_1
\end{aligned} \tag{48}$$

$$\begin{aligned}
D_x \mathbf{h}_{\parallel} &= 4v\mathbf{h}_{\perp} + \text{Re}(\bar{\mathbf{v}}_1 \cdot \mathbf{h}_1 + \bar{\mathbf{v}}_2 \cdot \mathbf{h}_2) \\
D_x \mathbf{H}_{\parallel} &= \mathbf{v}_2 \wedge \mathbf{h}_1 - \mathbf{v}_1 \wedge \mathbf{h}_2.
\end{aligned} \tag{49}$$

From the action (244) and (246) of the subgroup  $U(n-2)$  contained in the equivalence group, we have the unitary transformations

$$\mathbf{v} \rightarrow \mathbf{v}, \quad \mathbf{v}_1 \rightarrow \mathbf{v}_1 \mathbf{G}^{-1}, \quad \mathbf{v}_2 \rightarrow \mathbf{v}_2 \mathbf{G}^{-1} \tag{50}$$

$$\boldsymbol{\omega} \rightarrow \boldsymbol{\omega}, \quad \boldsymbol{\omega}_1 \rightarrow \boldsymbol{\omega}_1 \mathbf{G}^{-1}, \quad \boldsymbol{\omega}_2 \rightarrow \boldsymbol{\omega}_2 \mathbf{G}^{-1} \tag{51}$$

$$\mathbf{h}_{\perp} \rightarrow \mathbf{h}_{\perp}, \quad \mathbf{h}_1 \rightarrow \mathbf{h}_1 \mathbf{G}^{-1}, \quad \mathbf{h}_2 \rightarrow \mathbf{h}_2 \mathbf{G}^{-1} \tag{52}$$

and

$$\mathbf{H}_{\parallel} \rightarrow \bar{\mathbf{G}} \mathbf{H}_{\parallel} \mathbf{G}^{-1} \tag{53}$$

$$\Theta \rightarrow \Theta, \quad \theta \rightarrow \theta, \quad \boldsymbol{\Theta} \rightarrow \boldsymbol{\Theta} \tag{54}$$

where

$$\mathbf{G} = \mathbf{C} + i\mathbf{D} \in U(n-2). \tag{55}$$

Hence we see that the equations (46)–(49) are manifestly invariant under this  $U(n-2)$  subgroup (55). To exhibit their invariance under the subgroup  $SU(2)$  in the equivalence group (27), we observe the action (245) of this subgroup is given by the unitary transformations

$$\mathbf{v} \rightarrow \mathbf{v}, \quad (\mathbf{v}_1, \mathbf{v}_2) \rightarrow (\mathbf{v}_1, \mathbf{v}_2) \mathbf{G}^{-1} \tag{56}$$

$$\boldsymbol{\omega} \rightarrow \boldsymbol{\omega}, \quad (\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) \rightarrow (\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) \mathbf{G}^{-1} \tag{57}$$

$$\mathbf{h}_{\perp} \rightarrow \mathbf{h}_{\perp}, \quad (\mathbf{h}_1, \mathbf{h}_2) \rightarrow (\mathbf{h}_1, \mathbf{h}_2) \mathbf{G}^{-1} \tag{58}$$

and

$$\begin{pmatrix} 0 & -\mathbf{H}_{\parallel} \\ \mathbf{H}_{\parallel} & 0 \end{pmatrix} \rightarrow \bar{\mathbf{G}} \begin{pmatrix} 0 & -\mathbf{H}_{\parallel} \\ \mathbf{H}_{\parallel} & 0 \end{pmatrix} \mathbf{G}^{-1} \tag{59}$$

$$\begin{pmatrix} i\theta & \Theta \\ -\bar{\Theta} & -i\theta \end{pmatrix} \rightarrow \mathbf{G} \begin{pmatrix} i\theta & \Theta \\ -\bar{\Theta} & -i\theta \end{pmatrix} \mathbf{G}^{-1} \tag{60}$$

$$\begin{pmatrix} \boldsymbol{\Theta} & 0 \\ 0 & \boldsymbol{\Theta} \end{pmatrix} \rightarrow \mathbf{G} \begin{pmatrix} \boldsymbol{\Theta} & 0 \\ 0 & \boldsymbol{\Theta} \end{pmatrix} \mathbf{G}^{-1} \tag{61}$$

where

$$\mathbf{G} = \begin{pmatrix} \cos \lambda + id_2 \lambda^{-1} \sin \lambda & (c + id_1) \lambda^{-1} \sin \lambda \\ -(c - id_1) \lambda^{-1} \sin \lambda & \cos \lambda - id_2 \lambda^{-1} \sin \lambda \end{pmatrix} \in SU(2) \tag{62}$$

with

$$\lambda^2 = c^2 + d_1^2 + d_2^2. \tag{63}$$

The form of this  $SU(2)$  subgroup action (56)–(61) suggests that we rewrite equations (46)–(49) by introducing matrix products of the variables  $(\mathbf{v}_1, \mathbf{v}_2)$ ,  $(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$ , and  $(\mathbf{h}_1, \mathbf{h}_2)$ , as follows.

**Lemma 2.1.** *Let*

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} * (\mathbf{c}, \mathbf{d}) \equiv \begin{pmatrix} \mathbf{a} * \mathbf{c} & \mathbf{a} * \mathbf{d} \\ \mathbf{b} * \mathbf{c} & \mathbf{b} * \mathbf{d} \end{pmatrix} \quad (64)$$

where  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{C}^{n-2}$  are complex vectors and  $*$  :  $\mathbb{C}^{n-2} \times \mathbb{C}^{n-2} \rightarrow \mathbb{C}^{n-2}$  is any bilinear vector product. If  $(\mathbf{a}, \mathbf{b}) \rightarrow (\mathbf{a}, \mathbf{b})\mathbf{G}^{-1}$  and  $(\mathbf{c}, \mathbf{d}) \rightarrow (\mathbf{c}, \mathbf{d})\mathbf{G}^{-1}$  for  $\mathbf{G} \in SU(2)$ , then

$$\begin{pmatrix} \bar{\mathbf{a}} \\ \bar{\mathbf{b}} \end{pmatrix} * (\mathbf{c}, \mathbf{d}) \rightarrow \mathbf{G} \left( \begin{pmatrix} \bar{\mathbf{a}} \\ \bar{\mathbf{b}} \end{pmatrix} * (\mathbf{c}, \mathbf{d}) \right) \mathbf{G}^{-1} \quad (65)$$

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} * (\mathbf{c}, \mathbf{d}) \rightarrow \bar{\mathbf{G}} \left( \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} * (\mathbf{c}, \mathbf{d}) \right) \mathbf{G}^{-1}. \quad (66)$$

Using matrix products (2.1) constructed with  $*$  =  $\cdot$ ,  $*$  =  $\wedge$ ,  $*$  =  $\odot$ , (i.e. matrix dot product, matrix antisymmetric product, matrix symmetric product), we find that the equations (46)–(49) can be expressed in the form

$$D_t \mathbf{v} = (1/\sqrt{\chi}) \mathbf{h}_\perp + \frac{1}{4} D_x \omega - \frac{1}{2} \text{Im}((\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2) \cdot (\omega_1, \omega_2)) \quad (67)$$

$$\begin{aligned} D_t(\mathbf{v}_1, \mathbf{v}_2) &= (1/\sqrt{\chi})(\mathbf{h}_1, \mathbf{h}_2) + D_x(\omega_1, \omega_2) + i\frac{1}{2}\omega(\mathbf{v}_1, \mathbf{v}_2) - i2\mathbf{v}(\omega_1, \omega_2) \\ &\quad + (\mathbf{v}_1, \mathbf{v}_2) \begin{pmatrix} i\theta + \Theta & \Theta \\ -\bar{\Theta} & -i\theta + \Theta \end{pmatrix} \end{aligned} \quad (68)$$

$$\omega = \sqrt{\chi}(D_x \mathbf{h}_\perp + 4\mathbf{v} \mathbf{h}_\parallel - \text{Im}((\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2) \cdot (\mathbf{h}_1, \mathbf{h}_2))) \quad (69)$$

$$\begin{aligned} (\omega_1, \omega_2) &= \sqrt{\chi} \left( D_x(\mathbf{h}_1, \mathbf{h}_2) + (\mathbf{h}_\parallel + i\mathbf{h}_\perp)(\mathbf{v}_1, \mathbf{v}_2) + i2\mathbf{v}(\mathbf{h}_1, \mathbf{h}_2) \right. \\ &\quad \left. + (\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2) \begin{pmatrix} 0 & -\mathbf{H}_\parallel \\ \mathbf{H}_\parallel & 0 \end{pmatrix} \right) \end{aligned} \quad (70)$$

$$D_x \begin{pmatrix} i\theta & \Theta \\ -\bar{\Theta} & -i\theta \end{pmatrix} = \text{trfr} \left( \begin{pmatrix} \bar{\mathbf{v}}_1 \\ \bar{\mathbf{v}}_2 \end{pmatrix} \cdot (\omega_1, \omega_2) - \begin{pmatrix} \bar{\omega}_1 \\ \bar{\omega}_2 \end{pmatrix} \cdot (\mathbf{v}_1, \mathbf{v}_2) \right) \quad (71)$$

$$D_x \Theta = \text{tr}(\text{Re} \left( \begin{pmatrix} \bar{\mathbf{v}}_1 \\ \bar{\mathbf{v}}_2 \end{pmatrix} \wedge (\omega_1, \omega_2) \right) + i \text{Im} \left( \begin{pmatrix} \bar{\mathbf{v}}_1 \\ \bar{\mathbf{v}}_2 \end{pmatrix} \odot (\omega_1, \omega_2) \right)) \quad (72)$$

$$D_x \mathbf{h}_\parallel = 4\mathbf{v} \mathbf{h}_\perp + \text{Re}((\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2) \cdot (\mathbf{h}_1, \mathbf{h}_2)) \quad (73)$$

$$D_x \begin{pmatrix} 0 & -\mathbf{H}_\parallel \\ \mathbf{H}_\parallel & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} \wedge (\mathbf{h}_1, \mathbf{h}_2) + \begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \end{pmatrix} \wedge (\mathbf{v}_1, \mathbf{v}_2) \quad (74)$$

where  $\text{tr}$  and  $\text{trfr}$  respectively denote the trace of the matrix and the trace-free part of the matrix. As a consequence of Lemma 2.1 and the  $SU(2)$  group action (56)–(57), these equations (67)–(74) are manifestly invariant under the  $SU(2)$  subgroup (62) in the equivalence group, which establishes  $SU(2)$  invariance of equations (46)–(49).

We now eliminate the variables  $\Theta, \theta, \Theta$  through equation (48), and the variables  $\mathbf{h}_\parallel, \mathbf{H}_\parallel$  through equation (49). General results in Ref. [8] then establish the following result.

**Theorem 2.1.** *The flow equations (46)–(47) for the real scalar variable  $\mathbf{v}(t, x) \in \mathbb{R}$  and the pair of complex vector variables  $\mathbf{v}_1(t, x), \mathbf{v}_2(t, x) \in \mathbb{C}^{n-2}$  have the  $SU(2) \times U(n-2)$ -invariant form*

$$\begin{pmatrix} \mathbf{v} \\ \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}_t = \mathcal{H} \begin{pmatrix} \omega \\ \omega_1 \\ \omega_2 \end{pmatrix} + \frac{1}{\sqrt{\chi}} \begin{pmatrix} \mathbf{h}_\perp \\ \mathbf{h}_1 \\ \mathbf{h}_2 \end{pmatrix} \quad (75)$$

$$\begin{pmatrix} \omega \\ \omega_1 \\ \omega_2 \end{pmatrix} = \sqrt{\chi} \mathcal{J} \begin{pmatrix} h_\perp \\ \mathbf{h}_1 \\ \mathbf{h}_2 \end{pmatrix} \quad (76)$$

where

$$\mathcal{H} = \begin{pmatrix} \frac{1}{4}D_x & -\frac{1}{2}\text{Im}(\bar{\mathbf{v}}_1 \cdot) & -\frac{1}{2}\text{Im}(\bar{\mathbf{v}}_2 \cdot) \\ D_x - i2\mathbf{v} + \mathbf{v}_2 D_x^{-1}(\bar{\mathbf{v}}_2 \cdot) & -\mathbf{v}_2 D_x^{-1}(\mathbf{v}_1 \cdot \mathcal{C}) & \\ i\frac{1}{2}\mathbf{v}_1 & + \mathbf{v}_1 \rfloor D_x^{-1}(\text{Re}(\bar{\mathbf{v}}_1 \wedge) + i\text{Im}(\bar{\mathbf{v}}_1 \odot)) & + \mathbf{v}_1 \rfloor D_x^{-1}(\text{Re}(\bar{\mathbf{v}}_2 \wedge) + i\text{Im}(\bar{\mathbf{v}}_2 \odot)) \\ + i\mathbf{v}_1 D_x^{-1}(\text{Im}(\bar{\mathbf{v}}_1 \cdot)) & & - i\mathbf{v}_1 D_x^{-1}(\text{Im}(\bar{\mathbf{v}}_2 \cdot)) \\ i\frac{1}{2}\mathbf{v}_2 & - \mathbf{v}_1 D_x^{-1}(\mathbf{v}_2 \cdot \mathcal{C}) & D_x - i2\mathbf{v} + \mathbf{v}_1 D_x^{-1}(\bar{\mathbf{v}}_1 \cdot) \\ + \mathbf{v}_2 \rfloor D_x^{-1}(\text{Re}(\bar{\mathbf{v}}_1 \wedge) + i\text{Im}(\bar{\mathbf{v}}_1 \odot)) & + \mathbf{v}_2 \rfloor D_x^{-1}(\text{Re}(\bar{\mathbf{v}}_2 \wedge) + i\text{Im}(\bar{\mathbf{v}}_2 \odot)) & \\ - i\mathbf{v}_2 D_x^{-1}(\text{Im}(\bar{\mathbf{v}}_1 \cdot)) & & + i\mathbf{v}_2 D_x^{-1}(\text{Im}(\bar{\mathbf{v}}_2 \cdot)) \end{pmatrix} \quad (77)$$

and

$$\mathcal{J} = \begin{pmatrix} D_x + 16\mathbf{v} D_x^{-1}(\mathbf{v}) & -\text{Im}(\bar{\mathbf{v}}_1 \cdot) + 4\mathbf{v} D_x^{-1}(\text{Re}(\bar{\mathbf{v}}_1 \cdot)) & -\text{Im}(\bar{\mathbf{v}}_2 \cdot) + 4\mathbf{v} D_x^{-1}(\text{Re}(\bar{\mathbf{v}}_2 \cdot)) \\ i\mathbf{v}_1 + \mathbf{v}_1 D_x^{-1}(4\mathbf{v}) & D_x + i2\mathbf{v} + \mathbf{v}_1 D_x^{-1}(\text{Re}(\bar{\mathbf{v}}_1 \cdot)) & \mathbf{v}_1 D_x^{-1}(\text{Re}(\bar{\mathbf{v}}_2 \cdot)) \\ & + \bar{\mathbf{v}}_2 \rfloor D_x^{-1}(\mathbf{v}_2 \wedge) & - \bar{\mathbf{v}}_2 \rfloor D_x^{-1}(\mathbf{v}_1 \wedge) \\ i\mathbf{v}_2 + \mathbf{v}_2 D_x^{-1}(4\mathbf{v}) & \mathbf{v}_2 D_x^{-1}(\text{Re}(\bar{\mathbf{v}}_1 \cdot)) & D_x + i2\mathbf{v} + \mathbf{v}_2 D_x^{-1}(\text{Re}(\bar{\mathbf{v}}_2 \cdot)) \\ & - \bar{\mathbf{v}}_1 \rfloor D_x^{-1}(\mathbf{v}_2 \wedge) & + \bar{\mathbf{v}}_1 \rfloor D_x^{-1}(\mathbf{v}_1 \wedge) \end{pmatrix} \quad (78)$$

are compatible Hamiltonian cosymplectic and symplectic operators [12] on the  $x$ -jet space of  $(\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2)$ . Here  $\wedge$  and  $\odot$  are the outer products (37), and  $\mathcal{C}$  is the complex conjugation operator.

We remark that the counterpart of Theorem 2.1 in the flat Riemannian geometry  $\mathfrak{so}(2n)/\mathfrak{u}(n) \simeq \mathbb{R}^{n(n-1)}$  differs only the absence of the term  $1\sqrt{\chi}(h_\perp, \mathbf{h}_1, \mathbf{h}_2)$  in the operator equation (75). This is the only term contributed by the curvature of the space  $SO(2n)/U(n)$ . In particular, the Hamiltonian operators (77) and (78) are unchanged.

**2.2.  $U(n-2) \times SU(2)$ -invariant Hamiltonian operators.** We now explain some details about the Hamiltonian structure arising in Theorem 2.1.

For  $\mathbf{a}, \mathbf{b} \in \mathbb{R}$ ,  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2 \in \mathbb{C}^{n-2}$ , let

$$Q\left(\begin{pmatrix} \mathbf{a} \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{b} \\ \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}\right) = \mathbf{a}\mathbf{b} + \text{Re}(\bar{\mathbf{a}}_1 \cdot \mathbf{b}_1 + \bar{\mathbf{a}}_2 \cdot \mathbf{b}_2) \quad (79)$$

which denotes a Hermitian inner product determined by (up to an overall factor) the Cartan-Killing inner product (239) on the vector space  $\mathfrak{m}_\perp \simeq \mathbb{R} \oplus \mathbb{C}^{n-2} \oplus \mathbb{C}^{n-2}$  via the identification  $(\mathbf{a}, (\text{Re } \mathbf{a}_1, \text{Im } \mathbf{a}_1, \text{Re } \mathbf{a}_2, \text{Im } \mathbf{a}_2)) \in \mathfrak{m}_\perp$ . This inner product (79) is manifestly invariant under the group actions  $\mathbf{a} \rightarrow \mathbf{a}, \mathbf{a}_1 \rightarrow \mathbf{a}_1 \mathbf{G}^{-1}, \mathbf{a}_2 \rightarrow \mathbf{a}_2 \mathbf{G}^{-1}$  for  $\mathbf{G} \in U(n-2)$ , and

$\mathbf{a} \rightarrow \mathbf{a}, (\mathbf{a}_1, \mathbf{a}_1) \rightarrow (\mathbf{a}_1, \mathbf{a}_1)\mathbf{G}^{-1}$  for  $\mathbf{G} \in SU(2)$ , corresponding to the equivalence group (27), (55), (62). Let  $J^\infty$  denote the  $x$ -jet space of the variables  $(v, \mathbf{v}_1, \mathbf{v}_2)$ . Given a functional  $\mathfrak{H} = \int H(v, \mathbf{v}_1, \mathbf{v}_2, v_x, \mathbf{v}_{1x}, \mathbf{v}_{2x}, \dots) dx$  on  $J^\infty$ , we define the variational derivatives of  $\mathfrak{H}$  with respect to  $(v, \mathbf{v}_1, \mathbf{v}_2)$  by

$$X\mathfrak{H} = \int Q\left(\begin{pmatrix} \mathbf{a} \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix}, \begin{pmatrix} \delta\mathfrak{H}/\delta v \\ \delta\mathfrak{H}/\delta \mathbf{v}_1 \\ \delta\mathfrak{H}/\delta \mathbf{v}_2 \end{pmatrix}\right) dx \quad (80)$$

modulo boundary terms, where  $X = a\partial/\partial v + \text{Re}(\bar{\mathbf{a}}_1 \cdot \partial/\partial \mathbf{v}_1 + \bar{\mathbf{a}}_2 \cdot \partial/\partial \mathbf{v}_2)$  is an arbitrary vector field with components in  $\mathfrak{m}_\perp \simeq \mathbb{R} \oplus \mathbb{C}^{n-2} \oplus \mathbb{C}^{n-2}$  on  $J^\infty$ . Here  $(\delta/\delta v, \delta/\delta \mathbf{v}_1, \delta/\delta \mathbf{v}_2)$  denotes the standard Euler operator with respect to  $(v, \mathbf{v}_1, \mathbf{v}_2)$ .

Associated to the Hamiltonian operator  $\mathcal{H}$  is the Poisson bracket

$$\{\mathfrak{H}_1, \mathfrak{H}_2\}_{\mathcal{H}} := \int Q\left(\begin{pmatrix} \delta\mathfrak{H}_1/\delta v \\ \delta\mathfrak{H}_1/\delta \mathbf{v}_1 \\ \delta\mathfrak{H}_1/\delta \mathbf{v}_2 \end{pmatrix}, \mathcal{H}\begin{pmatrix} \delta\mathfrak{H}_2/\delta v \\ \delta\mathfrak{H}_2/\delta \mathbf{v}_1 \\ \delta\mathfrak{H}_2/\delta \mathbf{v}_2 \end{pmatrix}\right) dx \quad (81)$$

where  $\mathfrak{H}_1, \mathfrak{H}_2$  are real-valued functionals on  $J^\infty$ . The cosymplectic property of  $\mathcal{H}$  means that this bracket is skew-symmetric

$$\{\mathfrak{H}_1, \mathfrak{H}_2\}_{\mathcal{H}} + \{\mathfrak{H}_2, \mathfrak{H}_1\}_{\mathcal{H}} = 0 \quad (82)$$

and obeys the Jacobi identity

$$\{\mathfrak{H}_1, \{\mathfrak{H}_2, \mathfrak{H}_3\}_{\mathcal{H}}\}_{\mathcal{H}} + \text{cyclic} = 0. \quad (83)$$

A dual of the Poisson bracket is the symplectic 2-form associated to the operator  $\mathcal{J}$ ,

$$\omega(X_1, X_2)_{\mathcal{J}} := \int Q\left(\begin{pmatrix} \mathbf{a} \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix}, \mathcal{J}\begin{pmatrix} \mathbf{b} \\ \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}\right) dx \quad (84)$$

where  $X_1 = a\partial/\partial v + \text{Re}(\bar{\mathbf{a}}_1 \cdot \partial/\partial \mathbf{v}_1 + \bar{\mathbf{a}}_2 \cdot \partial/\partial \mathbf{v}_2)$ ,  $X_2 = b\partial/\partial v + \text{Re}(\bar{\mathbf{b}}_1 \cdot \partial/\partial \mathbf{v}_1 + \bar{\mathbf{b}}_2 \cdot \partial/\partial \mathbf{v}_2)$  are vector fields with components in  $\mathfrak{m}_\perp \simeq \mathbb{R} \oplus \mathbb{C}^{n-2} \oplus \mathbb{C}^{n-2}$  on  $J^\infty$ . The symplectic property of  $\mathcal{J}$  corresponds to  $\omega$  being skew-symmetric

$$\omega(X_1, X_2) + \omega(X_2, X_1) = 0 \quad (85)$$

and closed

$$\begin{aligned} & \text{pr}(X_1)\omega(X_2, X_3) + \text{cyclic} \\ &= Q\left(\begin{pmatrix} \mathbf{b} \\ \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}, \text{pr}(a\partial/\partial v + \text{Re}(\bar{\mathbf{a}}_1 \cdot \partial/\partial \mathbf{v}_1 + \bar{\mathbf{a}}_2 \cdot \partial/\partial \mathbf{v}_2))\mathcal{J}\begin{pmatrix} \mathbf{c} \\ \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix}\right) dx + \text{cyclic} = 0. \end{aligned} \quad (86)$$

Compatibility of the operators  $\mathcal{H}$  and  $\mathcal{J}$  is the statement that every linear combination  $c_1\mathcal{H} + c_2\mathcal{J}^{-1}$  is a cosymplectic Hamiltonian operator, or equivalently that  $c_1\mathcal{H}^{-1} + c_2\mathcal{J}$  is a symplectic operator, where  $\mathcal{H}^{-1}$  and  $\mathcal{J}^{-1}$  denote formal inverse operators defined on  $J^\infty$ .

### 3. NONLOCAL NLS SYSTEMS

For any non-stretching curve flow in  $M = SO(2n)/U(n)$  such that the tangent vector to the curve  $\gamma$  belongs to the algebraic equivalence class determined by the element (20) in  $\mathfrak{m} = \mathfrak{so}(2n)/\mathfrak{u}(n) \simeq T_\gamma M$ , Theorem 2.1 shows that the flow equation (75)–(76) for the components (39) of the connection matrix of the curve encodes a pair of compatible Hamiltonian operators (77)–(78). The flow itself is determined by freely specifying the perp components (23) of the evolution vector freely as a function of  $t$  at each point  $x$  along the curve. We emphasize that these Hamiltonian operators exist universally for all such flows.

A specific flow will have a Hamiltonian structure if the perp components (40) of the connection matrix along the curve flow have a variational form

$$\frac{1}{\sqrt{\chi}} \begin{pmatrix} \omega \\ \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \delta H / \delta \mathbf{v} \\ \delta H / \delta \mathbf{v}_1 \\ \delta H / \delta \mathbf{v}_2 \end{pmatrix} = \mathcal{J} \begin{pmatrix} \mathbf{h}_\perp \\ \mathbf{h}_1 \\ \mathbf{h}_2 \end{pmatrix} \quad (87)$$

for some Hamiltonian  $H$  given by a function of  $x, \mathbf{v}, \mathbf{v}_1, \mathbf{v}_2$ , and  $x$ -derivatives of  $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2$ , so then

$$\begin{pmatrix} \mathbf{v} \\ \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}_t - \frac{1}{\sqrt{\chi}} \begin{pmatrix} \mathbf{h}_\perp \\ \mathbf{h}_1 \\ \mathbf{h}_2 \end{pmatrix} = \mathcal{H} \begin{pmatrix} \delta H / \delta \mathbf{v} \\ \delta H / \delta \mathbf{v}_1 \\ \delta H / \delta \mathbf{v}_2 \end{pmatrix}. \quad (88)$$

Similarly, a specific Hamiltonian flow will have a bi-Hamiltonian structure if

$$\begin{pmatrix} \mathbf{v} \\ \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}_t - \frac{1}{\sqrt{\chi}} \begin{pmatrix} \mathbf{h}_\perp \\ \mathbf{h}_1 \\ \mathbf{h}_2 \end{pmatrix} = \mathcal{J}^{-1} \begin{pmatrix} \delta E / \delta \mathbf{v} \\ \delta E / \delta \mathbf{v}_1 \\ \delta E / \delta \mathbf{v}_2 \end{pmatrix} \quad (89)$$

holds for some Hamiltonian  $E$  again given by a function of  $x, \mathbf{v}, \mathbf{v}_1, \mathbf{v}_2$ , and  $x$ -derivatives of  $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2$ . In this case, the perp components (40) of the connection matrix along the curve flow will satisfy

$$\frac{1}{\sqrt{\chi}} \mathcal{R}^* \begin{pmatrix} \omega \\ \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \delta E / \delta \mathbf{v} \\ \delta E / \delta \mathbf{v}_1 \\ \delta E / \delta \mathbf{v}_2 \end{pmatrix} = \mathcal{J} \mathcal{R} \begin{pmatrix} \mathbf{h}_\perp \\ \mathbf{h}_1 \\ \mathbf{h}_2 \end{pmatrix} \quad (90)$$

where  $\mathcal{R} = \mathcal{H}\mathcal{J}$  is a recursion operator and  $\mathcal{R}^* = \mathcal{J}\mathcal{H}$  is its adjoint. When such Hamiltonian structures (87) and (90) exist, a scaling argument [13, 8] can be used to show that the Hamiltonians are related to the flow by

$$H = -\frac{1}{2} D_x^{-1} Q \left( \begin{pmatrix} \mathbf{v} \\ \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}, \text{ad}(\mathbf{e})^2 \mathcal{R} \begin{pmatrix} \mathbf{h}_\perp \\ \mathbf{h}_1 \\ \mathbf{h}_2 \end{pmatrix} \right), \quad (91)$$

$$E = -\frac{1}{4} D_x^{-1} Q \left( \begin{pmatrix} \mathbf{v} \\ \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}, \text{ad}(\mathbf{e})^2 \mathcal{R}^2 \begin{pmatrix} \mathbf{h}_\perp \\ \mathbf{h}_1 \\ \mathbf{h}_2 \end{pmatrix} \right). \quad (92)$$

In general, bi-Hamiltonian flows can be derived by [14] choosing the perp components (23) of the evolution vector to be the generator of a symmetry of the operators  $\mathcal{H}$  and  $\mathcal{J}$ . We will

now show that nonlocal bi-Hamiltonian NLS flows arise from the use of a unitary symmetry

$$\begin{pmatrix} h_\perp \\ \mathbf{h}_1 \\ \mathbf{h}_2 \end{pmatrix} = \begin{pmatrix} J(v) \\ J(\mathbf{v}_1) \\ J(\mathbf{v}_2) \end{pmatrix} \quad (93)$$

where  $J$  is a generator for either a  $U(1)$  subgroup or a  $SU(2)$  subgroup of the equivalence group  $SU(2) \times U(n-2)$  as follows.

First, there is a natural  $U(1)$  diagonal subgroup in  $U(n-2)$ . Its Lie algebra is generated by

$$J = i \in \mathfrak{u}(1) = C(\mathfrak{u}(n-2)) \subset \mathfrak{h}_\parallel = \mathfrak{su}(2) \oplus \mathfrak{u}(n-2) \quad (94)$$

where  $C$  denotes the center. Second, there is a natural  $SU(2)$  subgroup whose Lie algebra is generated by

$$J = \begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix} \in \mathfrak{su}(2) \subset \mathfrak{h}_\parallel = \mathfrak{su}(2) \oplus \mathfrak{u}(n-2) \quad (95)$$

$$a \in \mathbb{R}, \quad b \in \mathbb{C}, \quad a^2 + |b|^2 = 1.$$

In both cases,  $J$  acts on  $\mathbf{m}_\perp$  via the Lie bracket (17), as given by

$$J(\mathbf{m}_\perp) := -\text{ad}(J)\mathbf{m}_\perp \quad (96)$$

with  $J$  normalized so that  $\text{ad}(J)^2$  has eigenvalues  $-1, 0$ .

**3.1.  $U(1)$  NLS flow.** Using the Lie bracket (229), we find that the action of the  $U(1)$  symmetry generator (94) on the variables  $(v, \mathbf{v}_1, \mathbf{v}_2)$  is given by

$$J(v) = 0, \quad J(\mathbf{v}_1) = i\mathbf{v}_1, \quad J(\mathbf{v}_2) = i\mathbf{v}_2. \quad (97)$$

Hence we consider the corresponding flow defined by

$$\begin{aligned} h_\perp &= J(v) = 0, \\ \mathbf{h}_1 &= J(\mathbf{v}_1) = i\mathbf{v}_1, \\ \mathbf{h}_2 &= J(\mathbf{v}_2) = i\mathbf{v}_2. \end{aligned} \quad (98)$$

From the operator equation (76), we obtain

$$\begin{aligned} (1/\sqrt{\chi})\omega &= -|\mathbf{v}_1|^2 - |\mathbf{v}_2|^2, \\ (1/\sqrt{\chi})\omega_1 &= i\mathbf{v}_{1x} - 2v\mathbf{v}_1 - i2\bar{\mathbf{v}}_2 \rfloor D_x^{-1}(\mathbf{v}_1 \wedge \mathbf{v}_2), \\ (1/\sqrt{\chi})\omega_2 &= i\mathbf{v}_{2x} - 2v\mathbf{v}_2 - i2\bar{\mathbf{v}}_1 \rfloor D_x^{-1}(\mathbf{v}_2 \wedge \mathbf{v}_1). \end{aligned} \quad (99)$$

Then the operator equation (75) yields

$$v_t = -\frac{1}{2}(|\mathbf{v}_1|^2 + |\mathbf{v}_2|^2 + |D_x^{-1}(\mathbf{v}_1 \wedge \mathbf{v}_2)|^2)_x, \quad (100)$$

$$\begin{aligned} \mathbf{v}_{1t} - (1/\sqrt{\chi})i\mathbf{v}_1 &= i\mathbf{v}_{1xx} + i(4v^2 + |\mathbf{v}_1|^2 + |\mathbf{v}_2|^2)\mathbf{v}_1 - 2v_x\mathbf{v}_1 \\ &\quad - (i2\bar{\mathbf{v}}_{2x} + 4v\bar{\mathbf{v}}_2 + i2\mathbf{v}_1 \rfloor (D_x^{-1}(\bar{\mathbf{v}}_1 \wedge \bar{\mathbf{v}}_2))) \rfloor D_x^{-1}(\mathbf{v}_1 \wedge \mathbf{v}_2), \end{aligned} \quad (101)$$

$$\begin{aligned} \mathbf{v}_{2t} - (1/\sqrt{\chi})i\mathbf{v}_2 &= i\mathbf{v}_{2xx} + i(4v^2 + |\mathbf{v}_2|^2 + |\mathbf{v}_1|^2)\mathbf{v}_2 - 2v_x\mathbf{v}_2 \\ &\quad - (i2\bar{\mathbf{v}}_{1x} + 4v\bar{\mathbf{v}}_1 + i2\mathbf{v}_2 \rfloor (D_x^{-1}(\bar{\mathbf{v}}_2 \wedge \bar{\mathbf{v}}_1))) \rfloor D_x^{-1}(\mathbf{v}_2 \wedge \mathbf{v}_1). \end{aligned} \quad (102)$$

This is a multi-component nonlocal NLS system with  $U(1)$  invariance. It has the bi-Hamiltonian structure

$$\begin{pmatrix} \mathbf{v}_t \\ \mathbf{v}_{1t} \\ \mathbf{v}_{2t} \end{pmatrix} - \frac{1}{\sqrt{\chi}} \begin{pmatrix} 0 \\ i\mathbf{v}_1 \\ i\mathbf{v}_2 \end{pmatrix} = \mathcal{H} \begin{pmatrix} \delta H / \delta \mathbf{v} \\ \delta H / \delta \mathbf{v}_1 \\ \delta H / \delta \mathbf{v}_2 \end{pmatrix} = \mathcal{J}^{-1} \begin{pmatrix} \delta E / \delta \mathbf{v} \\ \delta E / \delta \mathbf{v}_1 \\ \delta E / \delta \mathbf{v}_2 \end{pmatrix} \quad (103)$$

where the Hamiltonians are given by

$$H = -\frac{1}{2} \text{Im} (\bar{\mathbf{v}}_1 \cdot \mathbf{v}_{1x} + \bar{\mathbf{v}}_2 \cdot \mathbf{v}_{2x}) - v(|\mathbf{v}_1|^2 + |\mathbf{v}_2|^2) + \frac{1}{2} \text{Im} ((\mathbf{v}_1 \wedge \mathbf{v}_2) \cdot D_x^{-1}(\bar{\mathbf{v}}_1 \wedge \bar{\mathbf{v}}_2)) \quad (104)$$

and

$$\begin{aligned} E = & \frac{1}{2} \text{Im} (\bar{\mathbf{v}}_{1x} \cdot \mathbf{v}_{1xx} + \bar{\mathbf{v}}_{2x} \cdot \mathbf{v}_{2xx}) + v_x(|\mathbf{v}_1|^2 + |\mathbf{v}_2|^2)_x + v(|\mathbf{v}_{1x}|^2 + |\mathbf{v}_{2x}|^2) \\ & - \frac{1}{2}(4v^2 + |\mathbf{v}_1|^2 + 2|\mathbf{v}_2|^2) \text{Im} (\bar{\mathbf{v}}_1 \cdot \mathbf{v}_{1x}) - \frac{1}{2}(4v^2 + |\mathbf{v}_2|^2 + 2|\mathbf{v}_1|^2) \text{Im} (\bar{\mathbf{v}}_2 \cdot \mathbf{v}_{2x}) \\ & + \frac{1}{2} \text{Im} ((\bar{\mathbf{v}}_1 \cdot \mathbf{v}_2)(\mathbf{v}_{1x} \cdot \bar{\mathbf{v}}_2) + (\bar{\mathbf{v}}_2 \cdot \mathbf{v}_1)(\mathbf{v}_{2x} \cdot \bar{\mathbf{v}}_1)) + 6v(|\mathbf{v}_1|^2 |\mathbf{v}_2|^2 - |\bar{\mathbf{v}}_1 \cdot \mathbf{v}_2|^2) \\ & - v(|\mathbf{v}_1|^2 + |\mathbf{v}_2|^2)(4v^2 + |\mathbf{v}_1|^2 + |\mathbf{v}_2|^2) \\ & + 2v(|\bar{\mathbf{v}}_1| D_x^{-1}(\mathbf{v}_1 \wedge \mathbf{v}_2)|^2 + |\bar{\mathbf{v}}_2| D_x^{-1}(\mathbf{v}_1 \wedge \mathbf{v}_2)|^2) \\ & + 2v \text{Re} ((\bar{\mathbf{v}}_1 \wedge \bar{\mathbf{v}}_2)_x \cdot D_x^{-1}(\mathbf{v}_1 \wedge \mathbf{v}_2)) + \text{Im} ((\bar{\mathbf{v}}_{1x} \wedge \bar{\mathbf{v}}_{2x}) - 4v^2(\bar{\mathbf{v}}_1 \wedge \bar{\mathbf{v}}_2) \\ & + \frac{1}{2}(\mathbf{v}_1| D_x^{-1}(\bar{\mathbf{v}}_1 \wedge \bar{\mathbf{v}}_2)) \wedge (\mathbf{v}_2| D_x^{-1}(\bar{\mathbf{v}}_1 \wedge \bar{\mathbf{v}}_2))) \cdot D_x^{-1}(\mathbf{v}_1 \wedge \mathbf{v}_2) \end{aligned} \quad (105)$$

from expressions (91) and (92). Note that a time-dependent  $U(1)$  phase transformation

$$\begin{pmatrix} v \\ \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} \rightarrow \exp(Jt) \begin{pmatrix} v \\ \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} v \\ \exp(it)\mathbf{v}_1 \\ \exp(it)\mathbf{v}_2 \end{pmatrix} \quad (106)$$

can be used to absorb the phase term (on the left hand side) in the system (100)–(102). This phase term itself has a Hamiltonian structure,

$$\begin{pmatrix} 0 \\ i\mathbf{v}_1 \\ i\mathbf{v}_2 \end{pmatrix} = \mathcal{H} \begin{pmatrix} \delta K / \delta v \\ \delta K / \delta \mathbf{v}_1 \\ \delta K / \delta \mathbf{v}_2 \end{pmatrix} \quad (107)$$

with the Hamiltonian given by

$$K = 2v. \quad (108)$$

Consequently, the NLS system (100)–(102) has a third Hamiltonian structure

$$\begin{pmatrix} \mathbf{v}_t \\ \mathbf{v}_{1t} \\ \mathbf{v}_{2t} \end{pmatrix} - \frac{1}{\sqrt{\chi}} \begin{pmatrix} 0 \\ i\mathbf{v}_1 \\ i\mathbf{v}_2 \end{pmatrix} = \mathcal{E} \begin{pmatrix} \delta K / \delta v \\ \delta K / \delta \mathbf{v}_1 \\ \delta K / \delta \mathbf{v}_2 \end{pmatrix} \quad (109)$$

where  $\mathcal{E} = \mathcal{R}\mathcal{H}$  is a Hamiltonian operator which is compatible with both  $\mathcal{H}$  and  $\mathcal{J}^{-1}$ .

**3.2.  $SU(2)$  NLS flows.** From the Lie bracket (229), the action of the  $SU(2)$  symmetry generator (95) on the variables  $(v, \mathbf{v}_1, \mathbf{v}_2)$  is given by

$$J(v) = 0, \quad (J(\mathbf{v}_1), J(\mathbf{v}_2)) = (\mathbf{v}_1, \mathbf{v}_2) \begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix} \quad (110)$$

where  $J$  satisfies

$$\bar{J}^t = -J, \quad \text{tr}(J) = 0. \quad (111)$$

Hence we consider the corresponding flow defined by

$$\begin{aligned} \mathbf{h}_\perp &= \mathbf{J}(\mathbf{v}) = 0 \\ (\mathbf{h}_1, \mathbf{h}_2) &= (\mathbf{J}(\mathbf{v}_1), \mathbf{J}(\mathbf{v}_2)) = (\mathbf{v}_1, \mathbf{v}_2) \begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix} = (ia\mathbf{v}_1 - \bar{b}\mathbf{v}_2, b\mathbf{v}_1 - ia\mathbf{v}_2). \end{aligned} \quad (112)$$

The operator equation (76) yields

$$\begin{aligned} (1/\sqrt{\chi})\omega &= \text{itr}(LJ), \\ (1/\sqrt{\chi})(\omega_1, \omega_2) &= (\mathbf{v}_{1x}, \mathbf{v}_{2x})\mathbf{J} + i2\mathbf{v}(\mathbf{v}_1, \mathbf{v}_2)\mathbf{J}. \end{aligned} \quad (113)$$

Then from the operator equation (75), we get

$$\mathbf{v}_t = i\frac{1}{2}\text{tr}(L_x\mathbf{J}), \quad (114)$$

$$\begin{aligned} (\mathbf{v}_{1t}, \mathbf{v}_{2t}) - (1/\sqrt{\chi})(\mathbf{v}_1, \mathbf{v}_2)\mathbf{J} &= (\mathbf{v}_{1xx}, \mathbf{v}_{2xx})\mathbf{J} + (i2\mathbf{v}_x + 4\mathbf{v}^2)(\mathbf{v}_1, \mathbf{v}_2)\mathbf{J} \\ &\quad + (\mathbf{v}_1, \mathbf{v}_2)(\{\mathbf{J}, L\} + \frac{1}{2}[\mathbf{J}, L] - \text{tr}(LJ)) \\ &\quad + (\mathbf{v}_1, \mathbf{v}_2)D_x^{-1}(i2\mathbf{v}[L, \mathbf{J}] + \frac{1}{2}[M, \mathbf{J}]), \end{aligned} \quad (115)$$

in terms of the matrices

$$L = \begin{pmatrix} \bar{\mathbf{v}}_1 \\ \bar{\mathbf{v}}_2 \end{pmatrix} \cdot (\mathbf{v}_1, \mathbf{v}_2), \quad M = \begin{pmatrix} \bar{\mathbf{v}}_1 \\ \bar{\mathbf{v}}_2 \end{pmatrix} \cdot (\mathbf{v}_{1x}, \mathbf{v}_{2x}) - \begin{pmatrix} \bar{\mathbf{v}}_{1x} \\ \bar{\mathbf{v}}_{2x} \end{pmatrix} \cdot (\mathbf{v}_1, \mathbf{v}_2). \quad (116)$$

This is a multi-component nonlocal NLS system with  $SU(2)$  invariance. It has the bi-Hamiltonian structure

$$\begin{pmatrix} \mathbf{v}_t \\ \mathbf{v}_{1t} \\ \mathbf{v}_{2t} \end{pmatrix} - \frac{1}{\sqrt{\chi}} \begin{pmatrix} 0 \\ \mathbf{J}(\mathbf{v}_1) \\ \mathbf{J}(\mathbf{v}_2) \end{pmatrix} = \mathcal{H} \begin{pmatrix} \delta H / \delta \mathbf{v} \\ \delta H / \delta \mathbf{v}_1 \\ \delta H / \delta \mathbf{v}_2 \end{pmatrix} = \mathcal{J}^{-1} \begin{pmatrix} \delta E / \delta \mathbf{v} \\ \delta E / \delta \mathbf{v}_1 \\ \delta E / \delta \mathbf{v}_2 \end{pmatrix} \quad (117)$$

where, through expressions (91) and (92), the Hamiltonians are given by

$$H = \text{tr}((i\mathbf{v}L + \frac{1}{4}M)\mathbf{J}) \quad (118)$$

$$\begin{aligned} E &= -\frac{1}{4}\text{Re}(\text{tr}(P\mathbf{J})) + \mathbf{v}\text{Im}(\text{tr}(N\mathbf{J})) + \mathbf{v}_x\text{Im}(\text{tr}(L\mathbf{J}))_x \\ &\quad + \frac{3}{8}\text{Re}(\text{tr}(M\mathbf{J})\text{tr}(L)) - \frac{1}{8}\text{Re}(\text{tr}(L\mathbf{J})\text{tr}(M)) + \frac{3}{8}\text{Re}(\text{tr}(LJL_x)) \\ &\quad + \mathbf{v}^2\text{tr}(M\mathbf{J}) - 4\mathbf{v}^3\text{Im}(\text{tr}(L\mathbf{J})) - \mathbf{v}\text{Im}(\text{tr}(L^2\mathbf{J})) \\ &\quad + \frac{1}{16}\text{Re}(\text{tr}(MD_x^{-1}[M, \mathbf{J}])) - \frac{1}{2}\mathbf{v}\text{Im}(\text{tr}(LD_x^{-1}[M, \mathbf{J}])) \\ &\quad - \mathbf{v}\text{Re}(\text{tr}(LD_x^{-1}(\mathbf{v}[L, \mathbf{J}])) \end{aligned} \quad (119)$$

in terms of the matrices

$$N = \begin{pmatrix} \bar{\mathbf{v}}_{1x} \\ \bar{\mathbf{v}}_{2x} \end{pmatrix} \cdot (\mathbf{v}_{1x}, \mathbf{v}_{2x}), \quad P = \begin{pmatrix} \bar{\mathbf{v}}_{1x} \\ \bar{\mathbf{v}}_{2x} \end{pmatrix} \cdot (\mathbf{v}_{1xx}, \mathbf{v}_{2xx}) - \begin{pmatrix} \bar{\mathbf{v}}_{1xx} \\ \bar{\mathbf{v}}_{2xx} \end{pmatrix} \cdot (\mathbf{v}_{1x}, \mathbf{v}_{2x}). \quad (120)$$

Here we have used the matrix anti-commutator identities

$$\{M, \mathbf{J}\} = \text{tr}(M)\mathbf{J} + \text{tr}(M\mathbf{J})I, \quad (121)$$

$$\{L, \mathbf{J}\} = \text{tr}(L)\mathbf{J} + \text{tr}(L\mathbf{J})I. \quad (122)$$



A time-dependent  $SU(2)$  phase transformation

$$\begin{pmatrix} v \\ \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} \rightarrow \exp(Jt) \begin{pmatrix} v \\ \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} v \\ \exp\left(\begin{pmatrix} ia & -\bar{b} \\ b & -ia \end{pmatrix} t\right) \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} \end{pmatrix} \quad (123)$$

can be used to absorb the phase term (on the left hand side) in this NLS system (114)–(115). Unlike the  $U(1)$  case, this phase term does not have a Hamiltonian structure.

We note that  $J$  depends on parameters  $a \in \mathbb{R}$  and  $b \in \mathbb{C}$  subject to the condition  $a^2 + |b|^2 = 1$ . Different choices of these parameters yield different  $SU(2)$  NLS systems which are related to each other by the  $SU(2)$  group action (56)–(62). We will illustrate this structure by choosing  $a, b$  such that  $J$  is  $i$  times one of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (124)$$

3.2.1.  $\sigma_1$  NLS flow. The choice  $J = i\sigma_1$  yields

$$h_\perp = J(v) = 0, \quad \mathbf{h}_1 = J(\mathbf{v}_1) = i\mathbf{v}_2, \quad \mathbf{h}_2 = J(\mathbf{v}_2) = i\mathbf{v}_1. \quad (125)$$

Then we have

$$\begin{aligned} (1/\sqrt{\chi})\omega &= -2\operatorname{Re}(\mathbf{v}_1 \cdot \bar{\mathbf{v}}_2) \\ (1/\sqrt{\chi})\omega_1 &= i\mathbf{v}_{2x} - 2v\mathbf{v}_2 \\ (1/\sqrt{\chi})\omega_2 &= i\mathbf{v}_{1x} - 2v\mathbf{v}_1 \end{aligned} \quad (126)$$

and

$$v_t = -\operatorname{Re}(\mathbf{v}_1 \cdot \bar{\mathbf{v}}_2)_x, \quad (127)$$

$$\begin{aligned} \mathbf{v}_{1t} - (1/\sqrt{\chi})i\mathbf{v}_2 &= i\mathbf{v}_{2xx} - 2v_x\mathbf{v}_2 + i4v^2\mathbf{v}_2 \\ &\quad + i\frac{1}{2}(3|\mathbf{v}_1|^2 + |\mathbf{v}_2|^2)\mathbf{v}_2 - i\frac{1}{2}(\bar{\mathbf{v}}_1 \cdot \mathbf{v}_2)\mathbf{v}_1 \\ &\quad + i4D_x^{-1}(v\operatorname{Im}(\mathbf{v}_1 \cdot \bar{\mathbf{v}}_2))\mathbf{v}_1 + 2D_x^{-1}(v(|\mathbf{v}_1|^2 - |\mathbf{v}_2|^2))\mathbf{v}_2 \\ &\quad + iD_x^{-1}(\operatorname{Re}(\mathbf{v}_1 \cdot \bar{\mathbf{v}}_{2x} - \mathbf{v}_{1x} \cdot \bar{\mathbf{v}}_2))\mathbf{v}_1 \\ &\quad - D_x^{-1}(\operatorname{Im}(\mathbf{v}_1 \cdot \bar{\mathbf{v}}_{1x} + \mathbf{v}_{2x} \cdot \bar{\mathbf{v}}_2))\mathbf{v}_2, \end{aligned} \quad (128)$$

$$\begin{aligned} \mathbf{v}_{2t} - (1/\sqrt{\chi})i\mathbf{v}_1 &= i\mathbf{v}_{1xx} - 2v_x\mathbf{v}_1 + i4v^2\mathbf{v}_1 \\ &\quad + i\frac{1}{2}(3|\mathbf{v}_2|^2 + |\mathbf{v}_1|^2)\mathbf{v}_1 - i\frac{1}{2}(\bar{\mathbf{v}}_2 \cdot \mathbf{v}_1)\mathbf{v}_2 \\ &\quad + i4D_x^{-1}(v\operatorname{Im}(\mathbf{v}_2 \cdot \bar{\mathbf{v}}_1))\mathbf{v}_2 + 2D_x^{-1}(v(|\mathbf{v}_2|^2 - |\mathbf{v}_1|^2))\mathbf{v}_1 \\ &\quad + iD_x^{-1}(\operatorname{Re}(\mathbf{v}_2 \cdot \bar{\mathbf{v}}_{1x} - \mathbf{v}_{2x} \cdot \bar{\mathbf{v}}_1))\mathbf{v}_2 \\ &\quad - D_x^{-1}(\operatorname{Im}(\mathbf{v}_2 \cdot \bar{\mathbf{v}}_{2x} + \mathbf{v}_{1x} \cdot \bar{\mathbf{v}}_1))\mathbf{v}_1. \end{aligned} \quad (129)$$

This nonlocal  $SU(2)$  NLS system (127)–(129) has the bi-Hamiltonian structure

$$\begin{pmatrix} v_t \\ \mathbf{v}_{1t} \\ \mathbf{v}_{2t} \end{pmatrix} - \frac{1}{\sqrt{\chi}} \begin{pmatrix} 0 \\ i\mathbf{v}_2 \\ i\mathbf{v}_1 \end{pmatrix} = \mathcal{H} \begin{pmatrix} \delta H / \delta v \\ \delta H / \delta \mathbf{v}_1 \\ \delta H / \delta \mathbf{v}_2 \end{pmatrix} = \mathcal{J}^{-1} \begin{pmatrix} \delta E / \delta v \\ \delta E / \delta \mathbf{v}_1 \\ \delta E / \delta \mathbf{v}_2 \end{pmatrix} \quad (130)$$

where

$$H = \text{Im}(\mathbf{v}_1 \cdot \bar{\mathbf{v}}_{2x}) - 2v \text{Re}(\mathbf{v}_1 \cdot \bar{\mathbf{v}}_2) \quad (131)$$

$$\begin{aligned} E = & \frac{1}{2} \text{Im}(\bar{\mathbf{v}}_{1x} \cdot \mathbf{v}_{2xx} + \bar{\mathbf{v}}_{2x} \cdot \mathbf{v}_{1xx}) + 2v \text{Re}(\bar{\mathbf{v}}_{1x} \cdot \mathbf{v}_{2x}) + 2v_x \text{Re}(\bar{\mathbf{v}}_1 \cdot \mathbf{v}_2)_x \\ & - \frac{1}{4}(8v_2 + 3|\mathbf{v}_1|^2 + 3|\mathbf{v}_2|^2) \text{Im}(\bar{\mathbf{v}}_1 \cdot \mathbf{v}_{2x} + \bar{\mathbf{v}}_2 \cdot \mathbf{v}_{1x}) \\ & + \frac{3}{4}(|\mathbf{v}_1|^2 - |\mathbf{v}_2|^2) \text{Im}(\bar{\mathbf{v}}_1 \cdot \mathbf{v}_2)_x - 2v(4v^2 + |\mathbf{v}_1|^2 + |\mathbf{v}_2|^2) \text{Re}(\bar{\mathbf{v}}_1 \cdot \mathbf{v}_2) \\ & + \text{Re}(D_x^{-1}(\bar{\mathbf{v}}_{1x} \cdot \mathbf{v}_2)) \text{Im}(\bar{\mathbf{v}}_1 \cdot \mathbf{v}_{1x}) + \text{Re}(D_x^{-1}(\bar{\mathbf{v}}_1 \cdot \mathbf{v}_{2x})) \text{Im}(\bar{\mathbf{v}}_2 \cdot \mathbf{v}_{2x}) \\ & - v(|\mathbf{v}_1|^2 - |\mathbf{v}_2|^2) \text{Re}(D_x^{-1}(\bar{\mathbf{v}}_1 \cdot \mathbf{v}_{2x} - \bar{\mathbf{v}}_2 \cdot \mathbf{v}_{1x})) \\ & + 2 \text{Im}(\bar{\mathbf{v}}_1 \cdot \mathbf{v}_{1x} - \bar{\mathbf{v}}_2 \cdot \mathbf{v}_{2x}) \text{Im}(D_x^{-1}(v\bar{\mathbf{v}}_1 \cdot \mathbf{v}_2)) + 4v(|\mathbf{v}_1|^2 - |\mathbf{v}_2|^2) \text{Im}(D_x^{-1}(v\bar{\mathbf{v}}_1 \cdot \mathbf{v}_2)). \end{aligned} \quad (132)$$

3.2.2.  $\sigma_2$  NLS flow. The choice  $J = i\sigma_2$  yields

$$\mathbf{h}_\perp = J(v) = 0, \quad \mathbf{h}_1 = J(\mathbf{v}_1) = \mathbf{v}_2, \quad \mathbf{h}_2 = J(\mathbf{v}_2) = -\mathbf{v}_1. \quad (133)$$

Then we have

$$\begin{aligned} (1/\sqrt{\chi})\omega &= -2 \text{Im}(\mathbf{v}_1 \cdot \bar{\mathbf{v}}_2) \\ (1/\sqrt{\chi})\omega_1 &= -\mathbf{v}_{2x} - i2v\mathbf{v}_2 \\ (1/\sqrt{\chi})\omega_2 &= \mathbf{v}_{1x} + i2v\mathbf{v}_1 \end{aligned} \quad (134)$$

and

$$\mathbf{v}_t = -\text{Im}(\mathbf{v}_1 \cdot \bar{\mathbf{v}}_2)_x, \quad (135)$$

$$\begin{aligned} \mathbf{v}_{1t} + (1/\sqrt{\chi})\mathbf{v}_2 = & -\mathbf{v}_{2xx} - i2v_x\mathbf{v}_2 - 4v^2\mathbf{v}_2 - \frac{1}{2}(3|\mathbf{v}_1|^2 + |\mathbf{v}_2|^2)\mathbf{v}_2 + \text{Re}(\mathbf{v}_1 \cdot \bar{\mathbf{v}}_2)\mathbf{v}_1 \\ & - i4D_x^{-1}(v \text{Re}(\mathbf{v}_1 \cdot \bar{\mathbf{v}}_2))\mathbf{v}_1 + i2D_x^{-1}(v(|\mathbf{v}_1|^2 - |\mathbf{v}_2|^2))\mathbf{v}_2 \\ & + iD_x^{-1}(\text{Im}(\mathbf{v}_1 \cdot \bar{\mathbf{v}}_{2x} - \mathbf{v}_{1x} \cdot \bar{\mathbf{v}}_2))\mathbf{v}_1 - iD_x^{-1}(\text{Im}(\mathbf{v}_1 \cdot \bar{\mathbf{v}}_{1x} - \mathbf{v}_2 \cdot \bar{\mathbf{v}}_{2x}))\mathbf{v}_2, \end{aligned} \quad (136)$$

$$\begin{aligned} \mathbf{v}_{2t} + (1/\sqrt{\chi})\mathbf{v}_1 = & \mathbf{v}_{1xx} + i2v_x\mathbf{v}_1 + 4v^2\mathbf{v}_1 + \frac{1}{2}(3|\mathbf{v}_2|^2 + |\mathbf{v}_1|^2)\mathbf{v}_1 - \text{Re}(\mathbf{v}_2 \cdot \bar{\mathbf{v}}_1)\mathbf{v}_2 \\ & + i4D_x^{-1}(v \text{Re}(\mathbf{v}_2 \cdot \bar{\mathbf{v}}_1))\mathbf{v}_2 - i2D_x^{-1}(v(|\mathbf{v}_2|^2 - |\mathbf{v}_1|^2))\mathbf{v}_1 \\ & - iD_x^{-1}(\text{Im}(\mathbf{v}_2 \cdot \bar{\mathbf{v}}_{1x} - \mathbf{v}_{2x} \cdot \bar{\mathbf{v}}_1))\mathbf{v}_2 + iD_x^{-1}(\text{Im}(\mathbf{v}_2 \cdot \bar{\mathbf{v}}_{2x} - \mathbf{v}_1 \cdot \bar{\mathbf{v}}_{1x}))\mathbf{v}_1. \end{aligned} \quad (137)$$

This nonlocal  $SU(2)$  NLS system (135)–(137) has the bi-Hamiltonian structure

$$\begin{pmatrix} \mathbf{v}_t \\ \mathbf{v}_{1t} \\ \mathbf{v}_{2t} \end{pmatrix} - \frac{1}{\sqrt{\chi}} \begin{pmatrix} 0 \\ \mathbf{v}_2 \\ -\mathbf{v}_1 \end{pmatrix} = \mathcal{H} \begin{pmatrix} \delta H / \delta v \\ \delta H / \delta \mathbf{v}_1 \\ \delta H / \delta \mathbf{v}_2 \end{pmatrix} = \mathcal{J}^{-1} \begin{pmatrix} \delta E / \delta v \\ \delta E / \delta \mathbf{v}_1 \\ \delta E / \delta \mathbf{v}_2 \end{pmatrix} \quad (138)$$

where

$$H = \operatorname{Re}(\mathbf{v}_{1x} \cdot \bar{\mathbf{v}}_2) - 2v \operatorname{Im}(\mathbf{v}_1 \cdot \bar{\mathbf{v}}_2) \quad (139)$$

$$\begin{aligned} E = & \frac{1}{2} \operatorname{Re}(\bar{\mathbf{v}}_{1x} \cdot \mathbf{v}_{2xx} - \bar{\mathbf{v}}_{2x} \cdot \mathbf{v}_{1xx}) - 2v \operatorname{Im}(\bar{\mathbf{v}}_{1x} \cdot \mathbf{v}_{2x}) - 2v_x \operatorname{Im}(\bar{\mathbf{v}}_1 \cdot \mathbf{v}_2)_x \\ & - \frac{1}{4}(8v^2 + 3|\mathbf{v}_1|^2 + 3|\mathbf{v}_2|^2) \operatorname{Re}(\bar{\mathbf{v}}_1 \cdot \mathbf{v}_{2x} - \bar{\mathbf{v}}_2 \cdot \mathbf{v}_{1x}) \\ & + \frac{3}{4}(|\mathbf{v}_1|^2 - |\mathbf{v}_2|^2) \operatorname{Re}(\bar{\mathbf{v}}_1 \cdot \mathbf{v}_2)_x + 2v(4v^2 + |\mathbf{v}_1|^2 + |\mathbf{v}_2|^2) \operatorname{Im}(\bar{\mathbf{v}}_1 \cdot \mathbf{v}_2) \\ & - \operatorname{Im}(D_x^{-1}(\bar{\mathbf{v}}_{1x} \cdot \mathbf{v}_2)) \operatorname{Im}(\bar{\mathbf{v}}_1 \cdot \mathbf{v}_{1x}) - \operatorname{Im}(D_x^{-1}(\bar{\mathbf{v}}_1 \cdot \mathbf{v}_{2x})) \operatorname{Im}(\bar{\mathbf{v}}_2 \cdot \mathbf{v}_{2x}) \\ & + v(|\mathbf{v}_1|^2 - |\mathbf{v}_2|^2) \operatorname{Im}(D_x^{-1}(\bar{\mathbf{v}}_1 \cdot \mathbf{v}_{2x} + \bar{\mathbf{v}}_2 \cdot \mathbf{v}_{1x})) \\ & + 2 \operatorname{Im}(\bar{\mathbf{v}}_1 \cdot \mathbf{v}_{1x} - \bar{\mathbf{v}}_2 \cdot \mathbf{v}_{2x}) \operatorname{Re}(D_x^{-1}(v\bar{\mathbf{v}}_1 \cdot \mathbf{v}_2)) + 4v(|\mathbf{v}_1|^2 - |\mathbf{v}_2|^2) \operatorname{Re}(D_x^{-1}(v\bar{\mathbf{v}}_1 \cdot \mathbf{v}_2)). \end{aligned} \quad (140)$$

3.2.3.  $\sigma_3$  NLS flow. The choice  $J = i\sigma_3$  yields

$$\mathbf{h}_\perp = J(v) = 0, \quad \mathbf{h}_1 = J(\mathbf{v}_1) = i\mathbf{v}_1, \quad \mathbf{h}_2 = J(\mathbf{v}_2) = -i\mathbf{v}_2. \quad (141)$$

Then we have

$$\begin{aligned} (1/\sqrt{\chi})\omega &= -|\mathbf{v}_1|^2 + |\mathbf{v}_2|^2 \\ (1/\sqrt{\chi})\omega_1 &= i\mathbf{v}_{1x} - 2v\mathbf{v}_1 \\ (1/\sqrt{\chi})\omega_2 &= -i\mathbf{v}_{2x} + 2v\mathbf{v}_2 \end{aligned} \quad (142)$$

and

$$\mathbf{v}_t = -\frac{1}{2}(|\mathbf{v}_1|^2 - |\mathbf{v}_2|^2)_x, \quad (143)$$

$$\begin{aligned} \mathbf{v}_{1t} - (1/\sqrt{\chi})i\mathbf{v}_1 &= i\mathbf{v}_{1xx} - 2v_x\mathbf{v}_1 + i4v^2\mathbf{v}_1 + i(|\mathbf{v}_1|^2 + |\mathbf{v}_2|^2)\mathbf{v}_1 - i(\mathbf{v}_1 \cdot \bar{\mathbf{v}}_2)\mathbf{v}_2 \\ &\quad - 4D_x^{-1}(v(\mathbf{v}_1 \cdot \bar{\mathbf{v}}_2))\mathbf{v}_2 + iD_x^{-1}(\mathbf{v}_{1x} \cdot \bar{\mathbf{v}}_2 - \mathbf{v}_1 \cdot \bar{\mathbf{v}}_{2x})\mathbf{v}_2, \end{aligned} \quad (144)$$

$$\begin{aligned} \mathbf{v}_{2t} - (1/\sqrt{\chi})i\mathbf{v}_2 &= -i\mathbf{v}_{2xx} + 2v_x\mathbf{v}_2 - i4v^2\mathbf{v}_2 - i(|\mathbf{v}_2|^2 + |\mathbf{v}_1|^2)\mathbf{v}_2 + i(\mathbf{v}_2 \cdot \bar{\mathbf{v}}_1)\mathbf{v}_1 \\ &\quad + 4D_x^{-1}(v(\mathbf{v}_2 \cdot \bar{\mathbf{v}}_1))\mathbf{v}_1 - iD_x^{-1}(\mathbf{v}_{2x} \cdot \bar{\mathbf{v}}_1 - \mathbf{v}_2 \cdot \bar{\mathbf{v}}_{1x})\mathbf{v}_1. \end{aligned} \quad (145)$$

This nonlocal  $SU(2)$  NLS system (143)–(145) has the bi-Hamiltonian structure

$$\begin{pmatrix} \mathbf{v}_t \\ \mathbf{v}_{1t} \\ \mathbf{v}_{2t} \end{pmatrix} - \frac{1}{\sqrt{\chi}} \begin{pmatrix} 0 \\ i\mathbf{v}_1 \\ -i\mathbf{v}_2 \end{pmatrix} = \mathcal{H} \begin{pmatrix} \delta H / \delta v \\ \delta H / \delta \mathbf{v}_1 \\ \delta H / \delta \mathbf{v}_2 \end{pmatrix} = \mathcal{J}^{-1} \begin{pmatrix} \delta E / \delta v \\ \delta E / \delta \mathbf{v}_1 \\ \delta E / \delta \mathbf{v}_2 \end{pmatrix} \quad (146)$$

where

$$H = \frac{1}{2} \operatorname{Im}(\mathbf{v}_1 \cdot \bar{\mathbf{v}}_{1x} - \mathbf{v}_2 \cdot \bar{\mathbf{v}}_{2x}) - v(|\mathbf{v}_1|^2 - |\mathbf{v}_2|^2) \quad (147)$$

$$\begin{aligned} E = & \frac{1}{2} \operatorname{Im}(\bar{\mathbf{v}}_{1x} \cdot \mathbf{v}_{1xx} - \bar{\mathbf{v}}_{2x} \cdot \mathbf{v}_{2xx}) + v(|\mathbf{v}_{1x}|^2 - |\mathbf{v}_{2x}|^2) + v_x(|\mathbf{v}_1|^2 - |\mathbf{v}_2|^2)_x \\ & - \frac{1}{2}(4v^2 + |\mathbf{v}_1|^2 + 2|\mathbf{v}_2|^2) \operatorname{Im}(\bar{\mathbf{v}}_1 \cdot \mathbf{v}_{1x}) + \frac{1}{2}(4v^2 + 2|\mathbf{v}_1|^2 + |\mathbf{v}_2|^2) \operatorname{Im}(\bar{\mathbf{v}}_2 \cdot \mathbf{v}_{2x}) \\ & + \frac{3}{4} \operatorname{Im}((\mathbf{v}_1 \cdot \bar{\mathbf{v}}_2)_x(\bar{\mathbf{v}}_1 \cdot \mathbf{v}_2)) - v(|\mathbf{v}_1|^2(4v^2 + |\mathbf{v}_1|^2) - |\mathbf{v}_2|^2(4v^2 + |\mathbf{v}_2|^2)) \\ & + \frac{1}{4} \operatorname{Im}((\bar{\mathbf{v}}_{1x} \cdot \mathbf{v}_2 - \bar{\mathbf{v}}_1 \cdot \mathbf{v}_{2x})D_x^{-1}(\mathbf{v}_{1x} \cdot \bar{\mathbf{v}}_2 - \mathbf{v}_1 \cdot \bar{\mathbf{v}}_{2x})) \\ & + 2 \operatorname{Re}((\bar{\mathbf{v}}_{1x} \cdot \mathbf{v}_2 - \bar{\mathbf{v}}_1 \cdot \mathbf{v}_{2x})D_x^{-1}(v\bar{\mathbf{v}}_1 \cdot \bar{\mathbf{v}}_2)) + 4v \operatorname{Im}((\bar{\mathbf{v}}_1 \cdot \mathbf{v}_2)D_x^{-1}(v\bar{\mathbf{v}}_1 \cdot \bar{\mathbf{v}}_2)). \end{aligned} \quad (148)$$

#### 4. GEOMETRIC CURVE FLOWS

From Theorem 2.1, the Hamiltonian operator equations (75)–(76) yield a flow on  $v = (v, (\mathbf{v}_1, \mathbf{v}_2))$  once  $h_\perp = (h_\perp, (\mathbf{h}_1, \mathbf{h}_2))$  is specified as a function of  $x$ ,  $v$ , and  $x$ -derivatives of  $v$ . The resulting flow equations correspond to a non-stretching curve flow  $\gamma(t, x)$  in  $M = SO(2n)/U(n)$  formulated in terms of the  $U(n)$ -parallel frame (20)–(21), by using the frame transport equation (10) and the Cartan structure equations (12)–(15) for the linear coframe  $e$  and linear connection  $\omega$  defined along  $\gamma$ . In particular, the flow vector  $Y = \gamma_t$  has the frame components

$$e \lrcorner \gamma_t = h_\perp + h_\parallel \quad (149)$$

where  $h_\parallel = (h_\parallel, \mathbf{H}_\parallel)$  is given by the structure equation

$$D_x h_\parallel = [h_\perp, u]_\parallel \quad (150)$$

expressed in terms of

$$u = \sqrt{\chi} \text{ad}(e)v = \omega \lrcorner \gamma_x = (\mathbf{u}, (\mathbf{u}_1, \mathbf{u}_2)) \quad (151)$$

$$e = e \lrcorner \gamma_x = (1/\sqrt{\chi}, (\mathbf{0}, \mathbf{0})) \quad (152)$$

which are the frame components of the tangential part of the linear connection along the curve, and the tangent vector  $X = \gamma_x$  of the curve, respectively. These geometrical relations give the equation of motion

$$\gamma_t = Y = Q(e^*, h_\perp + h_\parallel) = Q(e^*, \mathcal{Y}(h_\perp)) \quad (153)$$

where  $e^*$  is the linear frame dual to the linear coframe  $e$ ,  $Q$  is the inner product (79), and  $\mathcal{Y}$  is the operator

$$\mathcal{Y} := \text{id} - D_x^{-1} \text{ad}(u)_\parallel. \quad (154)$$

Here both  $e^*$  and  $e$  are understood to be determined by  $\omega \lrcorner \gamma_x = \sqrt{\chi} \text{ad}(e)v$  from the transport equation (10) along  $\gamma$ , up to the action of the equivalence group (27) of the  $U(n)$ -parallel frame, under which  $e \lrcorner \gamma_x = e$  is preserved.

We remark that a curve flow equation (153) in  $M = SO(2n)/U(n)$  will be  $SO(2n)$ -invariant if and only if  $h_\perp$  is restricted to be an equivariant function of  $x$ ,  $v$ , and  $x$ -derivatives of  $v$  (or equivalently,  $x$ ,  $u$ , and  $x$ -derivatives of  $u$ ) under the frame equivalence group in  $U(n)$ .

We will now derive the curve flow equations corresponding to the nonlocal  $U(1)$  NLS flow (100)–(102) and the nonlocal  $SU(2)$  NLS flow (114)–(115). These non-stretching curve flows can be expressed geometrically in a  $SO(2n)$ -invariant form in terms of the tangent vector  $X = \gamma_x$  and the principal normal vector  $N = \nabla_x \gamma_x$  of the curve, plus the Riemannian metric tensor  $g(\cdot, \cdot)$  and the Riemannian curvature tensor  $R(\cdot, \cdot)$  on  $M = SO(2n)/U(n)$ .

To proceed, we will need to use the decomposition of the tangent space  $T_\gamma M$  corresponding to  $\mathfrak{m} = \mathfrak{m}_\perp \oplus \mathfrak{m}_\parallel$  as follows. We first note that a projection operator onto  $\mathfrak{m}_\perp$  in  $\mathfrak{m}$  can be constructed out of the linear map  $\text{ad}(e)^2$  whose null space is  $\mathfrak{m}_\parallel$ . When restricted to  $\mathfrak{m}_\perp$ , this linear map has two eigenspaces, with eigenvalues  $-4/\chi$  and  $-1/\chi$ , whence

$$\mathcal{P}_\perp = (-\chi^2/4)((5/\chi)\text{id} + \text{ad}(e)^2)\text{ad}(e)^2 \quad (155)$$

satisfies  $\mathcal{P}_\perp \mathfrak{m}_\perp = \mathfrak{m}_\perp$  and  $\mathcal{P}_\perp \mathfrak{m}_\parallel = 0$ . Then a projection operator onto  $\mathfrak{m}_\parallel$  in  $\mathfrak{m}$  is given by

$$\mathcal{P}_\parallel = \text{id} - \mathcal{P}_\perp \quad (156)$$

which satisfies  $\mathcal{P}_{\parallel}\mathbf{m}_{\parallel} = \mathbf{m}_{\parallel}$  and  $\mathcal{P}_{\parallel}\mathbf{m}_{\perp} = 0$ . The soldering identification  $T_{\gamma}M \simeq \mathfrak{m}$  provided by the linear coframe  $e$  thereby yields the decomposition

$$T_{\gamma}M = (T_{\gamma}M)_{\perp} \oplus (T_{\gamma}M)_{\parallel} \quad (157)$$

via  $e \rfloor Z_{\perp} = \mathcal{P}_{\perp}(e \rfloor Z)$  and  $e \rfloor Z_{\parallel} = \mathcal{P}_{\parallel}(e \rfloor Z)$ , for all  $Z \in T_{\gamma}M$ . Since the projection operators (155) and (156) depend only on  $e \in \mathfrak{m}_{\parallel}$ , they are invariant under the equivalence group of the  $U(n)$ -parallel frame. Hence, through the soldering relation (152), this decomposition of the tangent space (157) depends only on  $\gamma_x$ . There are corresponding projection operators onto the orthogonal subspaces  $(T_{\gamma}M)_{\perp}$  and  $(T_{\gamma}M)_{\parallel}$  which can be constructed out of the linear map  $\text{ad}^2(X) = \text{ad}^2(\gamma_x)$  defined from

$$\text{ad}^2(Z) = -R(\cdot, Z)Z \quad (158)$$

for all  $Z \in T_{\gamma}M$ . In particular, we have

$$e \rfloor \text{ad}^2(X)Z = -e \rfloor R(Z, X)X = [[e \rfloor Z, e \rfloor X], e \rfloor X] = \text{ad}(e)^2(e \rfloor Z) \quad (159)$$

through the soldering relation (8).

Next, we will need the frame components of the principle normal vector

$$e \rfloor N = e \rfloor \nabla_x \gamma_x = -\text{ad}(e)u = -\sqrt{\chi} \text{ad}(e)^2 v = \frac{1}{\sqrt{\chi}} (4v, (\mathbf{v}_1, \mathbf{v}_2)) \quad (160)$$

which come from the soldering relations (6) and (151) together with the property  $\partial_x e = 0$ . Note that  $v$  itself represents the frame components of a vector  $\tilde{N}$  which is related to  $N$  by

$$v = \frac{1}{\sqrt{\chi}} e \rfloor \tilde{N}, \quad \text{ad}(e)^2 v = \frac{1}{\sqrt{\chi}} e \rfloor (\text{ad}^2(X) \tilde{N}) \quad (161)$$

so thus

$$N = -\text{ad}^2(X) \tilde{N}. \quad (162)$$

**4.1.  $U(1)$  NLS curve flows.** The  $U(1)$  NLS flow (100)–(102) is given by

$$(\mathbf{h}_{\perp}, (\mathbf{h}_1, \mathbf{h}_2)) = (0, (i\mathbf{v}_1, i\mathbf{v}_2)) \in \mathfrak{m}_{\perp}. \quad (163)$$

The structure equation (29) then gives

$$(\mathbf{h}_{\parallel}, \mathbf{H}_{\parallel}) = (0, i2D_x^{-1}(\mathbf{v}_2 \wedge \mathbf{v}_1)) \in \mathfrak{m}_{\parallel}. \quad (164)$$

These expressions directly determine

$$e \rfloor (\gamma_t)_{\perp} = (0, (i\mathbf{v}_1, i\mathbf{v}_2)) \in \mathfrak{m}_{\perp}, \quad (165)$$

$$e \rfloor (\gamma_t)_{\parallel} = (0, i2D_x^{-1}(\mathbf{v}_2 \wedge \mathbf{v}_1)) \in \mathfrak{m}_{\parallel}. \quad (166)$$

To write  $e \rfloor (\gamma_t)_{\perp}$  and  $e \rfloor (\gamma_t)_{\parallel}$  in a geometrical form, we consider the action  $-\text{ad}(J)$  of the unitary subgroup  $U(1) = C(U(n-2))$  of the equivalence group  $SU(2) \times U(n-2) \subset U(n)$ , where  $J$  is its generator (94) and  $C$  denotes the center. Through the soldering identification provided by the linear coframe  $e$ , this unitary action corresponds to an almost complex structure  $J_{\gamma}(\cdot)$  defined in the tangent space  $T_{\gamma}M$  along the curve via

$$e \rfloor J_{\gamma}(Z) = -\text{ad}(J)(e \rfloor Z) \quad (167)$$

for all  $Z \in T_\gamma M$ . We note  $J_\gamma$  commutes with  $\text{ad}^2(\gamma_x)$ , so thus  $J_\gamma(Z_\perp)$  lies in  $(T_\gamma M)_\perp$  and  $J_\gamma(Z_\parallel)$  lies in  $(T_\gamma M)_\parallel$ , while  $J_\gamma^2$  has eigenvalues  $-1, 0$  on  $(T_\gamma M)_\perp$ . In particular, we have

$$-\text{ad}(J)(\mathbf{h}_\perp, (\mathbf{h}_1, \mathbf{h}_2)) = (0, (i\mathbf{h}_1, i\mathbf{h}_2)) \quad (168)$$

which shows that the almost complex structure  $J_\gamma$  on  $(T_\gamma M)_\perp$  is degenerate. (On  $(T_\gamma M)_\parallel$ ,  $J_\gamma$  acts as  $-\text{ad}(J)(\mathbf{h}_\parallel, \mathbf{H}_\parallel) = (0, i2\mathbf{H}_\parallel)$ , so thus  $J_\gamma^2$  has eigenvalues  $-4, 0$ .)

Now, from the frame components (160) of  $N = \nabla_x \gamma_x$ , we see that

$$(0, (i\mathbf{v}_1, i\mathbf{v}_2)) = -\text{ad}(J)(4\mathbf{v}, (\mathbf{v}_1, \mathbf{v}_2)) = -\sqrt{\chi}\text{ad}(J)e\rfloor N = \sqrt{\chi}e\rfloor J_\gamma(N). \quad (169)$$

Equating expressions (169) and (165), we find

$$(\gamma_t)_\perp = \sqrt{\chi}J_\gamma(N). \quad (170)$$

Next, consider the frame components of  $\text{ad}^2(N)X = \text{ad}^2(\nabla_x \gamma_x)\gamma_x$ . From the soldering relations (152) and (160), we obtain

$$\begin{aligned} e\rfloor(\text{ad}^2(N)X)_\parallel &= \chi[\text{ad}(e)^2v, [\text{ad}(e)^2v, e]]_\parallel \\ &= \frac{1}{\sqrt{\chi}^3}[(4\mathbf{v}, (\mathbf{v}_1, \mathbf{v}_2)), [(4\mathbf{v}, (\mathbf{v}_1, \mathbf{v}_2)), (1, (\mathbf{0}, \mathbf{0}))]]_\parallel \\ &= \frac{1}{\sqrt{\chi}^3}(-((8\mathbf{v})^2 + |\mathbf{v}_1|^2 + |\mathbf{v}_2|^2), 2(\mathbf{v}_1 \wedge \mathbf{v}_2)) \end{aligned} \quad (171)$$

using the Lie brackets (232) and (237). Hence, we have

$$\begin{aligned} 2(0, i(\mathbf{v}_2 \wedge \mathbf{v}_1)) &= -\text{ad}(J)(-((8\mathbf{v})^2 + |\mathbf{v}_1|^2 + |\mathbf{v}_2|^2), 2(\mathbf{v}_1 \wedge \mathbf{v}_2)) \\ &= \sqrt{\chi}^3\text{ad}(J)(e\rfloor(\text{ad}^2(N)X)_\parallel) \\ &= \sqrt{\chi}^3e\rfloor J_\gamma(\text{ad}^2(N)_\parallel X) \end{aligned} \quad (172)$$

which yields the  $x$ -derivative of expression (166). This gives

$$\begin{aligned} \sqrt{\chi}^3e\rfloor J_\gamma(\text{ad}^2(N)_\parallel X) &= \partial_x(e\rfloor(\gamma_t)_\parallel) = \nabla_x e\rfloor(\gamma_t)_\parallel + e\rfloor\nabla_x(\gamma_t)_\parallel \\ &= -\text{ad}(u)e\rfloor(\gamma_t)_\parallel + e\rfloor\nabla_x(\gamma_t)_\parallel \end{aligned} \quad (173)$$

using the transport equation (10). We now use the soldering relations (160) and (161) to write

$$u = \sqrt{\chi}\text{ad}(e)v = [e\rfloor X, e\rfloor\tilde{N}] \quad (174)$$

so thus

$$\text{ad}(u)e\rfloor(\gamma_t)_\parallel = [[e\rfloor X, e\rfloor\tilde{N}], e\rfloor(\gamma_t)_\parallel] = -e\rfloor R(X, \tilde{N})(\gamma_t)_\parallel. \quad (175)$$

Then the relation (173) becomes

$$e\rfloor\sqrt{\chi}^3J_\gamma(\text{ad}^2(N)_\parallel X) = e\rfloor(\nabla_x(\gamma_t)_\parallel + R(X, \tilde{N})(\gamma_t)_\parallel) \quad (176)$$

from which we obtain

$$(\gamma_t)_\parallel = \sqrt{\chi}^3(\nabla_x + R(X, \tilde{N}))^{-1}J_\gamma(\text{ad}^2(N)_\parallel X). \quad (177)$$

Finally, after combining the geometrical expressions (170) and (177), followed by rescaling  $t \rightarrow \sqrt{\chi}t$ , we get

$$\gamma_t = J_\gamma(\nabla_x \gamma_x) + \chi(\nabla_x - R(\gamma_x, \text{ad}^2(\gamma_x)^{-1}\nabla_x \gamma_x))^{-1}J_\gamma(\text{ad}^2(\nabla_x \gamma_x)_\parallel \gamma_x), \quad |\gamma_x|_g = 1. \quad (178)$$

This is a non-stretching curve flow equation for  $\gamma(t, x)$  on  $M = SO(2n)/U(n)$ . Its form resembles a nonlocal generalization of a Schrodinger map  $\gamma_t = J(\nabla_x \gamma_x)$ , where  $J$  denotes the complex structure tensor on  $M = SO(2n)/U(n)$ . Schrodinger maps do not locally preserve the arclength of the curve, whereas both the local and nonlocal parts in the curve flow equation (178) can be shown to preserve  $|\gamma_x|_g = 1$ , due to the properties of  $J_\gamma$ .

**4.2.  $SU(2)$  NLS curve flows.** The general  $SU(2)$  NLS flow (114)–(115) is given by

$$(\mathbf{h}_\perp, (\mathbf{h}_1, \mathbf{h}_2)) = (0, (\mathbf{v}_1, \mathbf{v}_2)J) \in \mathfrak{m}_\perp \quad (179)$$

where  $J$  is the  $2 \times 2$  matrix (95) generating the action  $-\text{ad}(J)$  of the unitary subgroup  $SU(2)$  of the equivalence group  $SU(2) \times U(n-2) \subset U(n)$ . Note  $J$  can be written as a linear combination of Pauli matrices (124),

$$J = a(i\sigma_3) - \text{Re } b(i\sigma_2) + \text{Im } b(i\sigma_1), \quad a^2 + |b|^2 = 1 \quad (180)$$

parameterized by  $a \in \mathbb{R}$  and  $b \in \mathbb{C}$ , where  $\frac{1}{2}i\sigma_1, \frac{1}{2}i\sigma_2, \frac{1}{2}i\sigma_3$  comprise a normalized basis for the Lie algebra  $\mathfrak{su}(2)$ .

For this flow (179) the structure equation (29) gives simply

$$(\mathbf{h}_\parallel, \mathbf{H}_\parallel) = (0, \mathbf{0}) \in \mathfrak{m}_\parallel \quad (181)$$

and hence the expressions (179) and (181) determine

$$e](\gamma_t)_\perp = (0, (\mathbf{v}_1, \mathbf{v}_2)J) \in \mathfrak{m}_\perp, \quad (182)$$

$$e](\gamma_t)_\parallel = (0, \mathbf{0}) \in \mathfrak{m}_\parallel. \quad (183)$$

To write  $e](\gamma_t)_\perp$  in a geometrical form, we express the unitary action  $-\text{ad}(J)$  as an almost complex structure  $J_\gamma(\cdot)$  defined in the tangent space  $T_\gamma M$  along the curve by the soldering identification (167), where  $J_\gamma$  commutes with  $\text{ad}^2(\gamma_x)$  and has eigenvalues  $-1, 0$  on  $(T_\gamma M)_\perp$ . In particular, here we have

$$-\text{ad}(J)(\mathbf{h}_\perp, (\mathbf{h}_1, \mathbf{h}_2)) = (0, (\mathbf{h}_1, \mathbf{h}_2)J), \quad (184)$$

so thus this almost complex structure is degenerate. (On  $(T_\gamma M)_\parallel$ ,  $J_\gamma$  acts trivially as  $-\text{ad}(J)(\mathbf{h}_\parallel, \mathbf{H}_\parallel) = (0, \mathbf{0})$ .)

We now see that

$$(0, (\mathbf{v}_1, \mathbf{v}_2)J) = -\text{ad}(J)(4v, (\mathbf{v}_1, \mathbf{v}_2)) = -\sqrt{\chi}\text{ad}(J)e]N = \sqrt{\chi}e]J_\gamma(N) \quad (185)$$

using the frame components (160) of  $N = \nabla_x \gamma_x$ . Equating expressions (185) and (182), we obtain

$$e](\gamma_t)_\perp = \sqrt{\chi}e]J_\gamma(N) \quad (186)$$

while

$$e](\gamma_t)_\parallel = 0. \quad (187)$$

Then by combining these geometrical expressions, followed by rescaling  $t \rightarrow \sqrt{\chi}t$ , we get

$$\gamma_t = J_\gamma(\nabla_x \gamma_x), \quad |\gamma_x|_g = 1. \quad (188)$$

This is a non-stretching curve flow equation for  $\gamma(t, x)$  on  $M = SO(2n)/U(n)$ . It has the same form as a Schrodinger map  $\gamma_t = J(\nabla_x \gamma_x)$ , but where  $J$  denotes the complex structure tensor on  $M = SO(2n)/U(n)$ . Schrodinger maps do not locally preserve the arclength of the curve, whereas the curve flow equation (188) does, similarly to the  $U(1)$  flow equation (178).

It is interesting to note that the degenerate almost complex structure  $J_\gamma$  can be expressed as a linear combination of linear maps

$$J_\gamma = a_1 i_\gamma + a_2 j_\gamma + a_3 k_\gamma, \quad a_1^2 + a_2^2 + a_3^2 = 1, \quad (189)$$

corresponding to the Pauli matrices via the soldering identifications

$$\begin{aligned} e \rfloor i_\gamma(Z) &= -\text{ad}(\mathbf{i} \sigma_3)(e \rfloor Z) \\ e \rfloor j_\gamma(Z) &= -\text{ad}(\mathbf{i} \sigma_2)(e \rfloor Z) \\ e \rfloor k_\gamma(Z) &= -\text{ad}(\mathbf{i} \sigma_1)(e \rfloor Z) \end{aligned} \quad (190)$$

for all  $Z \in T_\gamma M$ . Each of these linear maps (190) commutes with  $\text{ad}^2(\gamma_x)$  and acts on  $(T_\gamma M)_\perp$  by

$$\begin{aligned} e \rfloor i_\gamma(Z_\perp) &= (0, (\mathbf{z}_1, \mathbf{z}_2) \mathbf{i} \sigma_3) \\ e \rfloor j_\gamma(Z_\perp) &= (0, (\mathbf{z}_1, \mathbf{z}_2) \mathbf{i} \sigma_2) \\ e \rfloor k_\gamma(Z_\perp) &= (0, (\mathbf{z}_1, \mathbf{z}_2) \mathbf{i} \sigma_1) \end{aligned} \quad (191)$$

where  $e \rfloor Z_\perp = (z, (\mathbf{z}_1, \mathbf{z}_2)) \in \mathfrak{m}_\perp$ . This structure can be connected with the algebra of imaginary quaternions. To make the relationship precise, we will need to decompose  $(T_\gamma M)_\perp$  into an orthogonal sum of eigenspaces of  $\text{ad}^2(\gamma_x)$ , corresponding to the eigenspaces (248) of  $\mathfrak{m}_\perp$  under  $\text{ad}(e)^2$ . We introduce the projection operators

$$\mathcal{P}_\perp^\mathbb{C} = (-\chi^2/3)((4/\chi)\text{id} + \text{ad}(e)^2)\text{ad}(e)^2 \quad (192)$$

$$\mathcal{P}_\perp^\mathbb{R} = (\chi^2/12)((12/\chi)\text{id} + \text{ad}(e)^2)\text{ad}(e)^2 \quad (193)$$

satisfying

$$\mathcal{P}_\perp^\mathbb{C}(z, (\mathbf{z}_1, \mathbf{z}_2)) = (0, (\mathbf{z}_1, \mathbf{z}_2)), \quad \mathcal{P}_\perp^\mathbb{R}(z, (\mathbf{z}_1, \mathbf{z}_2)) = (z, (\mathbf{0}, \mathbf{0})) \quad (194)$$

with  $\mathcal{P}_\perp^\mathbb{C} + \mathcal{P}_\perp^\mathbb{R} = \mathcal{P}_\perp$ . Using the soldering identification  $T_\gamma M \simeq \mathfrak{m}$  provided by the linear coframe  $e$ , we have

$$(T_\gamma M)_\perp = (T_\gamma M)_\perp^\mathbb{C} \oplus (T_\gamma M)_\perp^\mathbb{R} \quad (195)$$

via  $e \rfloor Z_\perp^\mathbb{C} = \mathcal{P}_\perp^\mathbb{C}(e \rfloor Z_\perp)$  and  $e \rfloor Z_\perp^\mathbb{R} = \mathcal{P}_\perp^\mathbb{R}(e \rfloor Z_\perp)$ , for all  $Z_\perp \in (T_\gamma M)_\perp$ . Then the composition of the linear maps  $i_\gamma, j_\gamma, k_\gamma$  on  $(T_\gamma M)_\perp^\mathbb{C}$  gives

$$\begin{aligned} -(i_\gamma)^2 &= -(j_\gamma)^2 = -(k_\gamma)^2 = \text{id}_\perp^\mathbb{C} \\ i_\gamma j_\gamma &= k_\gamma, \quad j_\gamma k_\gamma = i_\gamma, \quad k_\gamma i_\gamma = j_\gamma \end{aligned} \quad (196)$$

as derived from the algebra of the Pauli matrices (with  $\mathbb{I}$  denoting the  $2 \times 2$  unit matrix)

$$\begin{aligned} (\mathbf{i} \sigma_3)^2 &= (\mathbf{i} \sigma_2)^2 = (\mathbf{i} \sigma_1)^2 = -\mathbb{I} \\ (\mathbf{i} \sigma_3)(\mathbf{i} \sigma_2) &= \mathbf{i} \sigma_1, \quad (\mathbf{i} \sigma_2)(\mathbf{i} \sigma_1) = \mathbf{i} \sigma_3, \quad (\mathbf{i} \sigma_1)(\mathbf{i} \sigma_3) = \mathbf{i} \sigma_2 \end{aligned} \quad (197)$$

which is isomorphic to the algebra of imaginary quaternions  $\mathbb{Q} = \text{span}(i, j, k)$ . (In particular, the isomorphism is given by  $\mathbf{i} \sigma_3 \leftrightarrow i, \mathbf{i} \sigma_2 \leftrightarrow j, \mathbf{i} \sigma_1 \leftrightarrow k$ , and  $\mathbb{I} \leftrightarrow 1$ .) This establishes that there is a degenerate almost quaternionic structure defined in the tangent space  $T_\gamma M$  along the curve, which directly underlies the geometrical formulation of the  $SU(2)$  curve flow equation (188).



## 5. CONCLUDING REMARKS

In the literature on integrable systems connected with symmetric spaces, there is well-known general algebraic construction [15] of integrable NLS equations in Hermitian symmetric spaces  $M = G/H$ . This construction is based on a Lax pair method and essentially uses only the Lie bracket structure of the associated Hermitian symmetric Lie algebra  $\mathfrak{g}/\mathfrak{h}$  and the complex structure  $J$  associated with the adjoint action of the  $\mathfrak{u}(1)$  subalgebra given by the center of the unitary Lie algebra  $\mathfrak{h}$ . The NLS equations constructed from the Lax pair by this method are strictly local, in contrast to the nonlocal NLS equations obtained from the geometric moving frame method in the present paper using the Hermitian space  $M = SO(2n)/U(n)$  as well as in previous work using low-dimensional Hermitian spaces.

Moreover, the essential algebraic structure used in our construction of nonlocal NLS equations is a degenerate almost complex structure which is defined using a parallel frame and which is connected with the subalgebras  $\mathfrak{u}(1)$  and  $\mathfrak{su}(2)$  in the Lie algebra of the unitary gauge group of the parallel frame.

For future work, the geometric construction in this paper will be extended as far as possible to general Riemannian symmetric spaces, and a geometric realization will be explored for the algebraic Lax pair construction of local NLS equations.

## APPENDIX A. ALGEBRAIC STRUCTURES

A general reference for the following material is Ref. [16, 8].

Recall, the complex special orthogonal group  $SO(n, \mathbb{C})$  is the group of matrices  $g$  in  $GL(n, \mathbb{C})$  that leaves invariant the quadratic form  $z_1^2 + \cdots + z_n^2$  in terms of coordinates  $(z_1, \dots, z_n) \in \mathbb{C}^n$ , i.e.

$$g^t g = I_n, \quad (198)$$

with  $I_n$  denoting the identity matrix in  $GL(n, \mathbb{C})$ . Also recall, the complex unitary group  $U(n)$  is the group of matrices  $g$  in  $GL(n, \mathbb{C})$  that leaves invariant the Hermitian form  $z_1 \bar{z}_1 + \cdots + z_n \bar{z}_n$ , i.e.

$$g^t \bar{g} = I_n. \quad (199)$$

There is an embedding of  $U(n)$  into  $SO(2n) = SO(2n, \mathbb{C}) \cap GL(n, \mathbb{R})$  defined by

$$g \in U(n) \mapsto \begin{pmatrix} \operatorname{Re} g & \operatorname{Im} g \\ -\operatorname{Im} g & \operatorname{Re} g \end{pmatrix} \in SO(2n). \quad (200)$$

For later convenience, we let  $\mathfrak{s}(n, \mathbb{C})$  denote the vector space of symmetric matrices  $g$  in  $\mathfrak{gl}(n, \mathbb{C})$ , i.e.  $g^t = g$ .

**A.1. Symmetric Lie algebra  $\mathfrak{so}(2n) \supset \mathfrak{u}(n)$ .** The special orthogonal Lie algebra  $\mathfrak{so}(2n)$  consists of all matrices  $g$  in  $\mathfrak{gl}(2n, \mathbb{R})$  satisfying

$$g + g^t = 0, \quad (201)$$

There is an involutive automorphism of  $\mathfrak{gl}(2n, \mathbb{R})$  given by

$$\sigma(g) = JgJ^{-1}, \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \quad (202)$$

preserving  $\mathfrak{so}(2n) \subset \mathfrak{gl}(2n, \mathbb{R})$ . The matrices in  $\mathfrak{so}(2n)$  that are invariant under  $\sigma$  span the compact unitary Lie algebra  $\mathfrak{u}(n)$ . This leads to the orthogonal decomposition of  $\mathfrak{g} = \mathfrak{so}(2n)$  as a symmetric Lie algebra given by the eigenspaces of  $\sigma$ ,

$$\mathfrak{h} := \mathfrak{u}(n) \subset \mathfrak{g}, \quad \sigma(\mathfrak{h}) = \mathfrak{h} \quad (203)$$

and

$$\mathfrak{m} := \mathfrak{so}(2n)/\mathfrak{u}(n) \subset \mathfrak{g}, \quad \sigma(\mathfrak{m}) = -\mathfrak{m}. \quad (204)$$

These eigenspaces have the Lie bracket relations

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}. \quad (205)$$

**Lemma A.1.**

1. The matrix representation of the Lie algebra  $\mathfrak{g} = \mathfrak{so}(2n)$  is given by

$$\begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} \in \mathfrak{so}(2n), \quad A^t = -A, \quad C^t = -C, \quad (206)$$

where  $A, B, C \in \mathfrak{gl}(n, \mathbb{R})$ . The matrix representations of the vector space  $\mathfrak{m} = \mathfrak{so}(2n)/\mathfrak{u}(n)$  and the Lie subalgebra  $\mathfrak{h} = \mathfrak{u}(n)$  in  $\mathfrak{gl}(2n, \mathbb{R})$  are respectively given by

$$(A, B) := \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in \mathfrak{m}, \quad A^t = -A, \quad B^t = -B \quad (207)$$

$$(C, D) := \begin{pmatrix} C & D \\ -D & C \end{pmatrix} \in \mathfrak{h}, \quad C^t = -C, \quad D^t = D \quad (208)$$

where  $A, B, C, D \in \mathfrak{gl}(n, \mathbb{R})$ .

2. The Lie bracket relations (205) have the matrix representation

$$[\mathfrak{h}, \mathfrak{h}] = [(C, D), (R, S)] = (CR - RC - DS + SD, CS - SC + DR - RD) \in \mathfrak{h} \quad (209)$$

$$[\mathfrak{m}, \mathfrak{h}] = [(A, B), (C, D)] = (AC - CA - BD - DB, AD + DA + BC - CB) \in \mathfrak{m} \quad (210)$$

$$[\mathfrak{m}, \mathfrak{m}] = [(A, B), (P, Q)] = (AP - PA + BQ - QB, AQ + QA - BP - PB) \in \mathfrak{h}. \quad (211)$$

3. The restriction of the Cartan-Killing form on  $\mathfrak{g} = \mathfrak{so}(2n)$  to  $\mathfrak{m} = \mathfrak{so}(2n)/\mathfrak{u}(n)$  yields a negative-definite inner product

$$\langle \mathfrak{m}, \mathfrak{m} \rangle = \langle (A, B), (P, Q) \rangle = 4(n-1)\text{tr}(AP + BQ). \quad (212)$$

4. The (real) dimension of  $\mathfrak{m} = \mathfrak{so}(2n)/\mathfrak{u}(n)$  is  $n(n-1)$  and its rank is  $\lfloor n/2 \rfloor$ .

Consider the subspace  $\mathfrak{a} \subset \mathfrak{m} = \mathfrak{so}(2n)/\mathfrak{u}(n)$  spanned by the matrices

$$\begin{pmatrix} E_k & 0 \\ 0 & -E_k \end{pmatrix} \in \mathfrak{m}, \quad E_k = \mathbf{e}_{2k-1}^t \wedge \mathbf{e}_{2k}, \quad k = 1, \dots, \lfloor n/2 \rfloor \quad (213)$$

where

$$\mathbf{e}_{2k-1} = (\underbrace{0, \dots, 0}_{k-1}, 1, \underbrace{0, \dots, 0}_{n-k}) \in \mathbb{R}^n, \quad \mathbf{e}_{2k} = (\underbrace{0, \dots, 0}_k, 1, \underbrace{0, \dots, 0}_{n-k-1}) \in \mathbb{R}^n. \quad (214)$$

This is a Cartan subspace [17], which has dimension  $\lfloor n/2 \rfloor$ . For any choice of an element  $\mathbf{e} \in \mathfrak{a}$ , the corresponding linear operator  $\text{ad}(\mathbf{e})$  induces a direct sum decomposition of the vector spaces  $\mathfrak{m} = \mathfrak{so}(2n)/\mathfrak{u}(n)$  and  $\mathfrak{h} = \mathfrak{u}(n)$  into centralizer spaces  $\mathfrak{m}_{\parallel}$  and  $\mathfrak{h}_{\parallel}$  and their orthogonal complements (perp spaces)  $\mathfrak{m}_{\perp}$  and  $\mathfrak{h}_{\perp}$  with respect to the Cartan-Killing form. These subspaces have the Lie bracket relations (16)–(19). From the Lie bracket relation (18),

$\mathfrak{h}_\perp$  is mapped into  $\mathfrak{m}_\perp$ , and vice versa, under  $\text{ad}(\mathbf{e})$ . Hence  $\text{ad}(\mathbf{e})^2$  defines a linear mapping of each subspace  $\mathfrak{h}_\perp$  and  $\mathfrak{m}_\perp$  into itself. It is convenient to normalize  $\mathbf{e}$  with respect to the Cartan-Killing form, so that  $-1 = \langle \mathbf{e}, \mathbf{e} \rangle$ .

There are two special choices of an element  $\mathbf{e} \in \mathfrak{a}$  having an interesting structure for the linear transformation group  $H_\parallel^* \subset H^* = \text{Ad}(H)$  that preserves  $\mathbf{e}$ , where  $H_\parallel^*$  is generated by Lie subalgebra  $\mathfrak{h}_\parallel \subset \mathfrak{h} = \mathfrak{u}(n)$ . When  $n \geq 2$  is even, the element

$$\mathbf{e} := \frac{1}{\sqrt{\chi}} \begin{pmatrix} E_1 + \cdots + E_{n/2} & 0 \\ 0 & -E_1 - \cdots - E_{n/2} \end{pmatrix} \in \mathfrak{a}, \quad -1 = \langle \mathbf{e}, \mathbf{e} \rangle = -4n(n-1)/\chi \quad (215)$$

can be shown to yield a symplectic group  $H_\parallel^* \simeq SU(2) \subset U(n)$ . For all  $n \geq 2$ , the element

$$\mathbf{e} := \frac{1}{\sqrt{\chi}} \begin{pmatrix} E_1 & 0 \\ 0 & -E_1 \end{pmatrix} \in \mathfrak{a}, \quad -1 = \langle \mathbf{e}, \mathbf{e} \rangle = -8(n-1)/\chi \quad (216)$$

yields a unitary group  $H_\parallel^* \simeq SU(2) \times U(n-2)$ , as follows.

**Lemma A.2.** *Let  $\mathbf{e}$  be the element (216) in  $\mathfrak{a}$ .*

1. *The matrix representations of  $\mathfrak{m}_\parallel$  and  $\mathfrak{m}_\perp$  in  $\mathfrak{m} = \mathfrak{so}(2n)/\mathfrak{u}(n)$  are given by*

$$(\mathbf{a}_\parallel, (\mathbf{A}_\parallel, \mathbf{B}_\parallel)) := \begin{pmatrix} A_\parallel & B_\parallel \\ B_\parallel & -A_\parallel \end{pmatrix} \in \mathfrak{m}_\parallel \quad (217)$$

$$(\mathbf{b}_\perp, (\mathbf{a}_{1\perp}, \mathbf{b}_{1\perp}, \mathbf{a}_{2\perp}, \mathbf{b}_{2\perp})) := \begin{pmatrix} A_\perp & B_\perp \\ B_\perp & -A_\perp \end{pmatrix} \in \mathfrak{m}_\perp \quad (218)$$

in which

$$A_\parallel = \begin{pmatrix} 0 & \mathbf{a}_\parallel & 0 \\ -\mathbf{a}_\parallel & 0 & 0 \\ 0 & 0 & \mathbf{A}_\parallel \end{pmatrix}, \quad B_\parallel = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{B}_\parallel \end{pmatrix}, \quad A_\parallel^t = -A_\parallel, \quad B_\parallel^t = -B_\parallel \quad (219)$$

$$A_\perp = \begin{pmatrix} 0 & 0 & \mathbf{a}_{1\perp} \\ 0 & 0 & \mathbf{a}_{2\perp} \\ -\mathbf{a}_{1\perp}^t & -\mathbf{a}_{2\perp}^t & 0 \end{pmatrix}, \quad B_\perp = \begin{pmatrix} 0 & \mathbf{b}_\perp & \mathbf{b}_{1\perp} \\ -\mathbf{b}_\perp & 0 & \mathbf{b}_{2\perp} \\ -\mathbf{b}_{1\perp}^t & -\mathbf{b}_{2\perp}^t & 0 \end{pmatrix}, \quad A_\perp^t = -A_\perp, \quad B_\perp^t = -B_\perp \quad (220)$$

where  $\mathbf{a}_\parallel, \mathbf{b}_\perp \in \mathbb{R}$ ,  $\mathbf{a}_{1\perp}, \mathbf{a}_{2\perp}, \mathbf{b}_{1\perp}, \mathbf{b}_{2\perp} \in \mathbb{R}^{n-2}$ ,  $\mathbf{A}_\parallel, \mathbf{B}_\parallel \in \mathfrak{so}(n-2)$ .

2. *The matrix representations of  $\mathfrak{h}_\parallel$  and  $\mathfrak{h}_\perp$  in  $\mathfrak{h} = \mathfrak{u}(n)$  are given by*

$$((\mathbf{c}_\parallel, \mathbf{d}_{1\parallel}, \mathbf{d}_{2\parallel}), (\mathbf{C}_\parallel, \mathbf{D}_\parallel)) := \begin{pmatrix} C_\parallel & D_\parallel \\ -D_\parallel & C_\parallel \end{pmatrix} \in \mathfrak{h}_\parallel, \quad (221)$$

$$(\mathbf{d}_\perp, (\mathbf{c}_{1\perp}, \mathbf{d}_{1\perp}, \mathbf{c}_{2\perp}, \mathbf{d}_{2\perp})) := \begin{pmatrix} C_\perp & D_\perp \\ -D_\perp & C_\perp \end{pmatrix} \in \mathfrak{h}_\perp, \quad (222)$$

in which

$$C_\parallel = \begin{pmatrix} 0 & \mathbf{c}_\parallel & 0 \\ -\mathbf{c}_\parallel & 0 & 0 \\ 0 & 0 & \mathbf{C}_\parallel \end{pmatrix}, \quad D_\parallel = \begin{pmatrix} \mathbf{d}_{2\parallel} & \mathbf{d}_{1\parallel} & 0 \\ \mathbf{d}_{1\parallel} & -\mathbf{d}_{2\parallel} & 0 \\ 0 & 0 & \mathbf{D}_\parallel \end{pmatrix}, \quad C_\parallel^t = -C_\parallel, \quad D_\parallel^t = D_\parallel \quad (223)$$

$$C_{\perp} = \begin{pmatrix} 0 & 0 & \mathbf{c}_{1\perp} \\ 0 & 0 & \mathbf{c}_{2\perp} \\ -\mathbf{c}_{1\perp}^t & -\mathbf{c}_{2\perp}^t & 0 \end{pmatrix}, \quad D_{\perp} = \begin{pmatrix} d_{\perp} & 0 & \mathbf{d}_{1\perp} \\ 0 & d_{\perp} & \mathbf{d}_{2\perp} \\ \mathbf{d}_{1\perp}^t & \mathbf{d}_{2\perp}^t & 0 \end{pmatrix}, \quad C_{\perp}^t = -C_{\perp}, \quad D_{\perp}^t = D_{\perp} \quad (224)$$

where  $c_{\parallel}, d_{1\parallel}, d_{2\parallel}, d_{\perp} \in \mathbb{R}$ ,  $\mathbf{c}_{1\perp}, \mathbf{c}_{2\perp}, \mathbf{d}_{1\perp}, \mathbf{d}_{2\perp} \in \mathbb{R}^{n-2}$ ,  $\mathbf{C}_{\parallel} \in \mathfrak{so}(n-2)$ ,  $\mathbf{D}_{\parallel} \in \mathfrak{s}(n-2)$ .

3.  $\dim \mathfrak{m}_{\parallel} = n^2 - 5n + 7$ ,  $\dim \mathfrak{m}_{\perp} = \dim \mathfrak{h}_{\perp} = 4n - 7$ ,  $\dim \mathfrak{h}_{\parallel} = n^2 - 5n + 9$ .

4. The linear operator  $\text{ad}(e)$  acts on  $\mathfrak{m}_{\perp}$  and  $\mathfrak{h}_{\perp}$  by

$$\begin{aligned} \text{ad}(e)(b_{\perp}, (\mathbf{a}_{1\perp}, \mathbf{b}_{1\perp}, \mathbf{a}_{2\perp}, \mathbf{b}_{2\perp})) &= \frac{1}{\sqrt{\chi}}(-2b_{\perp}, (\mathbf{a}_{2\perp}, \mathbf{b}_{2\perp}, -\mathbf{a}_{1\perp}, -\mathbf{b}_{1\perp})) \in \mathfrak{h}_{\perp}, \\ \text{ad}(e)(d_{\perp}, (\mathbf{c}_{1\perp}, \mathbf{d}_{1\perp}, \mathbf{c}_{2\perp}, \mathbf{d}_{2\perp})) &= \frac{1}{\sqrt{\chi}}(2d_{\perp}, (\mathbf{c}_{2\perp}, \mathbf{d}_{2\perp}, -\mathbf{c}_{1\perp}, -\mathbf{d}_{1\perp})) \in \mathfrak{m}_{\perp}. \end{aligned} \quad (225)$$

**Proposition A.1.**

1. The Lie brackets (16)–(18) are given by

$$\begin{aligned} [\mathfrak{m}_{\parallel}, \mathfrak{m}_{\parallel}] &= [(a_{\parallel}, (\mathbf{A}_{\parallel}, \mathbf{B}_{\parallel})), (p_{\parallel}, (\mathbf{P}_{\parallel}, \mathbf{Q}_{\parallel}))] \\ &= ((0, 0, 0), (\mathbf{A}_{\parallel}\mathbf{P}_{\parallel} - \mathbf{P}_{\parallel}\mathbf{A}_{\parallel} + \mathbf{B}_{\parallel}\mathbf{Q}_{\parallel} - \mathbf{Q}_{\parallel}\mathbf{B}_{\parallel}, \mathbf{A}_{\parallel}\mathbf{Q}_{\parallel} + \mathbf{Q}_{\parallel}\mathbf{A}_{\parallel} - \mathbf{B}_{\parallel}\mathbf{P}_{\parallel} - \mathbf{P}_{\parallel}\mathbf{B}_{\parallel})) \in \mathfrak{h}_{\parallel} \end{aligned} \quad (226)$$

$$\begin{aligned} [\mathfrak{m}_{\parallel}, \mathfrak{h}_{\parallel}] &= [(p_{\parallel}, (\mathbf{P}_{\parallel}, \mathbf{Q}_{\parallel})), ((c_{\parallel}, d_{1\parallel}, d_{2\parallel}), (\mathbf{C}_{\parallel}, \mathbf{D}_{\parallel}))] \\ &= (0, (\mathbf{P}_{\parallel}\mathbf{C}_{\parallel} - \mathbf{C}_{\parallel}\mathbf{P}_{\parallel} - \mathbf{Q}_{\parallel}\mathbf{D}_{\parallel} - \mathbf{D}_{\parallel}\mathbf{Q}_{\parallel}, \mathbf{P}_{\parallel}\mathbf{D}_{\parallel} + \mathbf{D}_{\parallel}\mathbf{P}_{\parallel} + \mathbf{Q}_{\parallel}\mathbf{C}_{\parallel} - \mathbf{C}_{\parallel}\mathbf{Q}_{\parallel})) \in \mathfrak{m}_{\parallel} \end{aligned} \quad (227)$$

$$\begin{aligned} [\mathfrak{h}_{\parallel}, \mathfrak{h}_{\parallel}] &= [((c_{\parallel}, d_{1\parallel}, d_{2\parallel}), (\mathbf{C}_{\parallel}, \mathbf{D}_{\parallel})), ((r_{\parallel}, s_{1\parallel}, s_{2\parallel}), (\mathbf{R}_{\parallel}, \mathbf{S}_{\parallel}))] \\ &= ((2d_{1\parallel}s_{2\parallel} - 2d_{2\parallel}s_{1\parallel}, -2c_{\parallel}s_{2\parallel} + 2d_{2\parallel}r_{\parallel}, 2c_{\parallel}s_{1\parallel} - 2d_{1\parallel}r_{\parallel}), \\ &\quad (\mathbf{C}_{\parallel}\mathbf{R}_{\parallel} - \mathbf{R}_{\parallel}\mathbf{C}_{\parallel} - \mathbf{D}_{\parallel}\mathbf{S}_{\parallel} + \mathbf{S}_{\parallel}\mathbf{D}_{\parallel}, \mathbf{C}_{\parallel}\mathbf{S}_{\parallel} - \mathbf{S}_{\parallel}\mathbf{C}_{\parallel} + \mathbf{D}_{\parallel}\mathbf{R}_{\parallel} - \mathbf{R}_{\parallel}\mathbf{D}_{\parallel})) \in \mathfrak{h}_{\parallel} \end{aligned} \quad (228)$$

$$\begin{aligned} [\mathfrak{m}_{\perp}, \mathfrak{h}_{\parallel}] &= [(q_{\perp}, (\mathbf{p}_{1\perp}, \mathbf{q}_{1\perp}, \mathbf{p}_{2\perp}, \mathbf{q}_{2\perp})), ((c_{\parallel}, d_{1\parallel}, d_{2\parallel}), (\mathbf{C}_{\parallel}, \mathbf{D}_{\parallel}))] \\ &= (0, (-d_{2\parallel}\mathbf{q}_{1\perp} - c_{\parallel}\mathbf{p}_{2\perp} - d_{1\parallel}\mathbf{q}_{2\perp} + \mathbf{p}_{1\perp}\mathbf{C}_{\parallel} - \mathbf{q}_{1\perp}\mathbf{D}_{\parallel}, \\ &\quad d_{2\parallel}\mathbf{p}_{1\perp} + d_{1\parallel}\mathbf{p}_{2\perp} - c_{\parallel}\mathbf{q}_{2\perp} + \mathbf{q}_{1\perp}\mathbf{C}_{\parallel} + \mathbf{p}_{1\perp}\mathbf{D}_{\parallel}, \\ &\quad c_{\parallel}\mathbf{p}_{1\perp} - d_{1\parallel}\mathbf{q}_{1\perp} + d_{2\parallel}\mathbf{q}_{2\perp} + \mathbf{p}_{2\perp}\mathbf{C}_{\parallel} - \mathbf{q}_{2\perp}\mathbf{D}_{\parallel}, \\ &\quad d_{1\parallel}\mathbf{p}_{1\perp} + c_{\parallel}\mathbf{q}_{1\perp} - d_{2\parallel}\mathbf{p}_{2\perp} + \mathbf{q}_{2\perp}\mathbf{C}_{\parallel} + \mathbf{p}_{2\perp}\mathbf{D}_{\parallel})) \in \mathfrak{m}_{\perp} \end{aligned} \quad (229)$$

$$\begin{aligned} [\mathfrak{m}_{\parallel}, \mathfrak{h}_{\perp}] &= [(a_{\parallel}, (\mathbf{A}_{\parallel}, \mathbf{B}_{\parallel})), (s_{\perp}, (\mathbf{r}_{1\perp}, \mathbf{s}_{1\perp}, \mathbf{r}_{2\perp}, \mathbf{s}_{2\perp}))] \\ &= (2a_{\parallel}s_{\perp}, (a_{\parallel}\mathbf{r}_{2\perp} - \mathbf{r}_{1\perp}\mathbf{A}_{\parallel} - \mathbf{s}_{1\perp}\mathbf{B}_{\parallel}, a_{\parallel}\mathbf{s}_{2\perp} + \mathbf{s}_{1\perp}\mathbf{A}_{\parallel} - \mathbf{r}_{1\perp}\mathbf{B}_{\parallel}, \\ &\quad -a_{\parallel}\mathbf{r}_{1\perp} - \mathbf{r}_{2\perp}\mathbf{A}_{\parallel} - \mathbf{s}_{2\perp}\mathbf{B}_{\parallel}, -a_{\parallel}\mathbf{s}_{1\perp} + \mathbf{s}_{2\perp}\mathbf{A}_{\parallel} - \mathbf{r}_{2\perp}\mathbf{B}_{\parallel})) \in \mathfrak{m}_{\perp} \end{aligned} \quad (230)$$

$$\begin{aligned} [\mathfrak{h}_{\parallel}, \mathfrak{h}_{\perp}] &= [((c_{\parallel}, d_{1\parallel}, d_{2\parallel}), (\mathbf{C}_{\parallel}, \mathbf{D}_{\parallel})), (s_{\perp}, (\mathbf{r}_{1\perp}, \mathbf{s}_{1\perp}, \mathbf{r}_{2\perp}, \mathbf{s}_{2\perp}))] \\ &= (0, (-d_{2\parallel}\mathbf{s}_{1\perp} + c_{\parallel}\mathbf{r}_{2\perp} - d_{1\parallel}\mathbf{s}_{2\perp} - \mathbf{r}_{1\perp}\mathbf{C}_{\parallel} + \mathbf{s}_{1\perp}\mathbf{D}_{\parallel}, \\ &\quad d_{2\parallel}\mathbf{r}_{1\perp} + d_{1\parallel}\mathbf{r}_{2\perp} + c_{\parallel}\mathbf{s}_{2\perp} - \mathbf{s}_{1\perp}\mathbf{C}_{\parallel} - \mathbf{r}_{1\perp}\mathbf{D}_{\parallel}, \\ &\quad -c_{\parallel}\mathbf{r}_{1\perp} - d_{1\parallel}\mathbf{s}_{1\perp} + d_{2\parallel}\mathbf{s}_{2\perp} - \mathbf{r}_{2\perp}\mathbf{C}_{\parallel} + \mathbf{s}_{2\perp}\mathbf{D}_{\parallel}, \\ &\quad d_{1\parallel}\mathbf{r}_{1\perp} - c_{\parallel}\mathbf{s}_{1\perp} - d_{2\parallel}\mathbf{r}_{2\perp} - \mathbf{s}_{2\perp}\mathbf{C}_{\parallel} - \mathbf{r}_{2\perp}\mathbf{D}_{\parallel})) \in \mathfrak{h}_{\perp} \end{aligned} \quad (231)$$

$$\begin{aligned}
[\mathbf{m}_{\parallel}, \mathbf{m}_{\perp}] &= [(\mathbf{a}_{\parallel}, (\mathbf{A}_{\parallel}, \mathbf{B}_{\parallel})), (\mathbf{q}_{\perp}, (\mathbf{p}_{1\perp}, \mathbf{q}_{1\perp}, \mathbf{p}_{2\perp}, \mathbf{q}_{2\perp}))] \\
&= (-2\mathbf{a}_{\parallel}\mathbf{q}_{\perp}, (\mathbf{a}_{\parallel}\mathbf{p}_{2\perp} - \mathbf{p}_{1\perp}\mathbf{A}_{\parallel} - \mathbf{q}_{1\perp}\mathbf{B}_{\parallel}, \mathbf{a}_{\parallel}\mathbf{q}_{2\perp} + \mathbf{q}_{1\perp}\mathbf{A}_{\parallel} - \mathbf{p}_{1\perp}\mathbf{B}_{\parallel}, \\
&\quad - \mathbf{a}_{\parallel}\mathbf{p}_{1\perp} - \mathbf{p}_{2\perp}\mathbf{A}_{\parallel} - \mathbf{q}_{2\perp}\mathbf{B}_{\parallel}, -\mathbf{a}_{\parallel}\mathbf{q}_{1\perp} + \mathbf{q}_{2\perp}\mathbf{A}_{\parallel} - \mathbf{p}_{2\perp}\mathbf{B}_{\parallel})) \in \mathfrak{h}_{\perp}.
\end{aligned} \tag{232}$$

2. The remaining Lie brackets (19) are given by

$$\begin{aligned}
[\mathbf{m}_{\perp}, \mathbf{m}_{\perp}]_{\mathfrak{h}_{\parallel}} &= [(\mathbf{b}_{\perp}, (\mathbf{a}_{1\perp}, \mathbf{b}_{1\perp}, \mathbf{a}_{2\perp}, \mathbf{b}_{2\perp})), (\mathbf{q}_{\perp}, (\mathbf{p}_{1\perp}, \mathbf{q}_{1\perp}, \mathbf{p}_{2\perp}, \mathbf{q}_{2\perp}))]_{\mathfrak{h}_{\parallel}} \\
&= ((-\mathbf{a}_{1\perp}\mathbf{p}_{2\perp}^t + \mathbf{a}_{2\perp}\mathbf{p}_{1\perp}^t - \mathbf{b}_{1\perp}\mathbf{q}_{2\perp}^t + \mathbf{b}_{2\perp}\mathbf{q}_{1\perp}^t, \\
&\quad - \mathbf{a}_{1\perp}\mathbf{q}_{2\perp}^t - \mathbf{a}_{2\perp}\mathbf{q}_{1\perp}^t + \mathbf{b}_{1\perp}\mathbf{p}_{2\perp}^t + \mathbf{b}_{2\perp}\mathbf{p}_{1\perp}^t, \\
&\quad - \mathbf{a}_{1\perp}\mathbf{q}_{1\perp}^t + \mathbf{a}_{2\perp}\mathbf{q}_{2\perp}^t + \mathbf{b}_{1\perp}\mathbf{p}_{1\perp}^t - \mathbf{b}_{2\perp}\mathbf{p}_{2\perp}^t), \\
&\quad (-\mathbf{a}_{1\perp}^t\mathbf{p}_{1\perp} + \mathbf{p}_{1\perp}^t\mathbf{a}_{1\perp} - \mathbf{a}_{2\perp}^t\mathbf{p}_{2\perp} + \mathbf{p}_{2\perp}^t\mathbf{a}_{2\perp} \\
&\quad - \mathbf{b}_{1\perp}^t\mathbf{q}_{1\perp} + \mathbf{q}_{1\perp}^t\mathbf{b}_{1\perp} - \mathbf{b}_{2\perp}^t\mathbf{q}_{2\perp} + \mathbf{q}_{2\perp}^t\mathbf{b}_{2\perp}, \\
&\quad - \mathbf{a}_{1\perp}^t\mathbf{q}_{1\perp} - \mathbf{q}_{1\perp}^t\mathbf{a}_{1\perp} - \mathbf{a}_{2\perp}^t\mathbf{q}_{2\perp} - \mathbf{q}_{2\perp}^t\mathbf{a}_{2\perp} \\
&\quad + \mathbf{b}_{1\perp}^t\mathbf{p}_{1\perp} + \mathbf{p}_{1\perp}^t\mathbf{b}_{1\perp} + \mathbf{b}_{2\perp}^t\mathbf{p}_{2\perp} + \mathbf{p}_{2\perp}^t\mathbf{b}_{2\perp})) \in \mathfrak{h}_{\parallel}
\end{aligned} \tag{233}$$

$$\begin{aligned}
[\mathbf{m}_{\perp}, \mathbf{m}_{\perp}]_{\mathfrak{h}_{\perp}} &= [(\mathbf{b}_{\perp}, (\mathbf{a}_{1\perp}, \mathbf{b}_{1\perp}, \mathbf{a}_{2\perp}, \mathbf{b}_{2\perp})), (\mathbf{q}_{\perp}, (\mathbf{p}_{1\perp}, \mathbf{q}_{1\perp}, \mathbf{p}_{2\perp}, \mathbf{q}_{2\perp}))]_{\mathfrak{h}_{\perp}} \\
&= (-\mathbf{a}_{1\perp}\mathbf{q}_{1\perp}^t - \mathbf{a}_{2\perp}\mathbf{q}_{2\perp}^t + \mathbf{b}_{1\perp}\mathbf{p}_{1\perp}^t + \mathbf{b}_{2\perp}\mathbf{p}_{2\perp}^t, \\
&\quad (-\mathbf{q}_{\perp}\mathbf{b}_{2\perp} + \mathbf{b}_{\perp}\mathbf{q}_{2\perp}, \mathbf{q}_{\perp}\mathbf{a}_{2\perp} - \mathbf{b}_{\perp}\mathbf{p}_{2\perp}, \mathbf{q}_{\perp}\mathbf{b}_{1\perp} - \mathbf{b}_{\perp}\mathbf{q}_{1\perp}, -\mathbf{q}_{\perp}\mathbf{a}_{1\perp} + \mathbf{b}_{\perp}\mathbf{p}_{1\perp})) \in \mathfrak{h}_{\perp}
\end{aligned} \tag{234}$$

$$\begin{aligned}
[\mathfrak{h}_{\perp}, \mathfrak{h}_{\perp}]_{\mathfrak{h}_{\parallel}} &= [(\mathbf{d}_{\perp}, (\mathbf{c}_{1\perp}, \mathbf{d}_{1\perp}, \mathbf{c}_{2\perp}, \mathbf{d}_{2\perp})), (\mathbf{s}_{\perp}, (\mathbf{r}_{1\perp}, \mathbf{s}_{1\perp}, \mathbf{r}_{2\perp}, \mathbf{s}_{2\perp}))]_{\mathfrak{h}_{\parallel}} \\
&= ((-\mathbf{c}_{1\perp}\mathbf{r}_{2\perp}^t + \mathbf{c}_{2\perp}\mathbf{r}_{1\perp}^t - \mathbf{d}_{1\perp}\mathbf{s}_{2\perp}^t + \mathbf{d}_{2\perp}\mathbf{s}_{1\perp}^t, \\
&\quad \mathbf{c}_{1\perp}\mathbf{s}_{2\perp}^t + \mathbf{c}_{2\perp}\mathbf{s}_{1\perp}^t - \mathbf{d}_{1\perp}\mathbf{r}_{2\perp}^t - \mathbf{d}_{2\perp}\mathbf{r}_{1\perp}^t, \\
&\quad \mathbf{c}_{1\perp}\mathbf{s}_{1\perp}^t - \mathbf{c}_{2\perp}\mathbf{s}_{2\perp}^t - \mathbf{r}_{1\perp}\mathbf{d}_{1\perp}^t + \mathbf{r}_{2\perp}\mathbf{d}_{2\perp}^t), \\
&\quad (-\mathbf{c}_{1\perp}^t\mathbf{r}_{1\perp} + \mathbf{r}_{1\perp}^t\mathbf{c}_{1\perp} - \mathbf{c}_{2\perp}^t\mathbf{r}_{2\perp} + \mathbf{r}_{2\perp}^t\mathbf{c}_{2\perp} \\
&\quad - \mathbf{d}_{1\perp}^t\mathbf{s}_{1\perp} + \mathbf{s}_{1\perp}^t\mathbf{d}_{1\perp} - \mathbf{d}_{2\perp}^t\mathbf{s}_{2\perp} + \mathbf{s}_{2\perp}^t\mathbf{d}_{2\perp}, \\
&\quad - \mathbf{c}_{1\perp}^t\mathbf{s}_{1\perp} - \mathbf{s}_{1\perp}^t\mathbf{c}_{1\perp} - \mathbf{c}_{2\perp}^t\mathbf{s}_{2\perp} - \mathbf{s}_{2\perp}^t\mathbf{c}_{2\perp} \\
&\quad + \mathbf{d}_{1\perp}^t\mathbf{r}_{1\perp} + \mathbf{r}_{1\perp}^t\mathbf{d}_{1\perp} + \mathbf{d}_{2\perp}^t\mathbf{r}_{2\perp} + \mathbf{r}_{2\perp}^t\mathbf{d}_{2\perp})) \in \mathfrak{h}_{\parallel}
\end{aligned} \tag{235}$$

$$\begin{aligned}
[\mathfrak{h}_{\perp}, \mathfrak{h}_{\perp}]_{\mathfrak{h}_{\perp}} &= [(\mathbf{d}_{\perp}, (\mathbf{c}_{1\perp}, \mathbf{d}_{1\perp}, \mathbf{c}_{2\perp}, \mathbf{d}_{2\perp})), (\mathbf{s}_{\perp}, (\mathbf{r}_{1\perp}, \mathbf{s}_{1\perp}, \mathbf{r}_{2\perp}, \mathbf{s}_{2\perp}))]_{\mathfrak{h}_{\perp}} \\
&= (\mathbf{c}_{1\perp}\mathbf{s}_{1\perp}^t + \mathbf{c}_{2\perp}\mathbf{s}_{2\perp}^t - \mathbf{d}_{1\perp}\mathbf{r}_{1\perp}^t - \mathbf{d}_{2\perp}\mathbf{r}_{2\perp}^t, \\
&\quad (\mathbf{s}_{\perp}\mathbf{d}_{1\perp} - \mathbf{d}_{\perp}\mathbf{s}_{1\perp}, -\mathbf{s}_{\perp}\mathbf{c}_{1\perp} + \mathbf{d}_{\perp}\mathbf{r}_{1\perp}, \mathbf{s}_{\perp}\mathbf{d}_{2\perp} - \mathbf{d}_{\perp}\mathbf{s}_{2\perp}, -\mathbf{s}_{\perp}\mathbf{c}_{2\perp} + \mathbf{d}_{\perp}\mathbf{r}_{2\perp})) \in \mathfrak{h}_{\perp}
\end{aligned} \tag{236}$$

$$\begin{aligned}
[\mathbf{m}_{\perp}, \mathfrak{h}_{\perp}]_{\mathfrak{m}_{\parallel}} &= [(\mathbf{b}_{\perp}, (\mathbf{a}_{1\perp}, \mathbf{b}_{1\perp}, \mathbf{a}_{2\perp}, \mathbf{b}_{2\perp})), (\mathbf{s}_{\perp}, (\mathbf{r}_{1\perp}, \mathbf{s}_{1\perp}, \mathbf{r}_{2\perp}, \mathbf{s}_{2\perp}))]_{\mathfrak{m}_{\parallel}} \\
&= (-2\mathbf{b}_{\perp}\mathbf{s}_{\perp} - \mathbf{a}_{1\perp}\mathbf{r}_{2\perp}^t + \mathbf{a}_{2\perp}\mathbf{r}_{1\perp}^t - \mathbf{b}_{1\perp}\mathbf{s}_{2\perp}^t + \mathbf{b}_{2\perp}\mathbf{s}_{1\perp}^t, \\
&\quad (-\mathbf{a}_{1\perp}^t\mathbf{r}_{1\perp} + \mathbf{r}_{1\perp}^t\mathbf{a}_{1\perp} - \mathbf{a}_{2\perp}^t\mathbf{r}_{2\perp} + \mathbf{r}_{2\perp}^t\mathbf{a}_{2\perp} \\
&\quad + \mathbf{b}_{1\perp}^t\mathbf{s}_{1\perp} - \mathbf{s}_{1\perp}^t\mathbf{b}_{1\perp} + \mathbf{b}_{2\perp}^t\mathbf{s}_{2\perp} - \mathbf{s}_{2\perp}^t\mathbf{b}_{2\perp}, \\
&\quad - \mathbf{a}_{1\perp}^t\mathbf{s}_{1\perp} + \mathbf{s}_{1\perp}^t\mathbf{a}_{1\perp} - \mathbf{a}_{2\perp}^t\mathbf{s}_{2\perp} + \mathbf{s}_{2\perp}^t\mathbf{a}_{2\perp} \\
&\quad - \mathbf{b}_{1\perp}^t\mathbf{r}_{1\perp} + \mathbf{r}_{1\perp}^t\mathbf{b}_{1\perp} - \mathbf{b}_{2\perp}^t\mathbf{r}_{2\perp} + \mathbf{r}_{2\perp}^t\mathbf{b}_{2\perp})) \in \mathfrak{m}_{\parallel}
\end{aligned} \tag{237}$$

$$\begin{aligned}
[\mathfrak{m}_\perp, \mathfrak{h}_\perp]_{\mathfrak{m}_\perp} &= [(\mathbf{b}_\perp, (\mathbf{a}_{1\perp}, \mathbf{b}_{1\perp}, \mathbf{a}_{2\perp}, \mathbf{b}_{2\perp})), (\mathbf{s}_\perp, (\mathbf{r}_{1\perp}, \mathbf{s}_{1\perp}, \mathbf{r}_{2\perp}, \mathbf{s}_{2\perp}))]_{\mathfrak{m}_\perp} \\
&= (\mathbf{a}_{1\perp} \mathbf{s}_{2\perp}^t - \mathbf{a}_{2\perp} \mathbf{s}_{1\perp}^t - \mathbf{b}_{1\perp} \mathbf{r}_{2\perp}^t + \mathbf{b}_{2\perp} \mathbf{r}_{1\perp}^t, \\
&\quad (-\mathbf{s}_\perp \mathbf{b}_{1\perp} - \mathbf{b}_\perp \mathbf{s}_{2\perp}, \mathbf{s}_\perp \mathbf{a}_{1\perp} + \mathbf{b}_\perp \mathbf{r}_{2\perp}, -\mathbf{s}_\perp \mathbf{b}_{2\perp} + \mathbf{b}_\perp \mathbf{s}_{1\perp}, \mathbf{s}_\perp \mathbf{a}_{2\perp} - \mathbf{b}_\perp \mathbf{r}_{1\perp})) \in \mathfrak{m}_\perp.
\end{aligned} \tag{238}$$

3. The Cartan-Killing form on  $\mathfrak{m}_\perp$  is given by

$$\begin{aligned}
&\langle (\mathbf{b}_\perp, (\mathbf{a}_{1\perp}, \mathbf{b}_{1\perp}, \mathbf{a}_{2\perp}, \mathbf{b}_{2\perp})), (\mathbf{q}_\perp, (\mathbf{p}_{1\perp}, \mathbf{q}_{1\perp}, \mathbf{p}_{2\perp}, \mathbf{q}_{2\perp})) \rangle \\
&= -8(n-1)(\mathbf{b}_\perp \mathbf{q}_\perp + \mathbf{a}_{1\perp} \mathbf{p}_{1\perp}^t + \mathbf{b}_{1\perp} \mathbf{q}_{1\perp}^t + \mathbf{a}_{2\perp} \mathbf{p}_{2\perp}^t + \mathbf{b}_{2\perp} \mathbf{q}_{2\perp}^t)
\end{aligned} \tag{239}$$

which is negative definite.

The adjoint action of the Lie subalgebra  $\mathfrak{h}_\parallel \subset \mathfrak{h} = \mathfrak{u}(n)$  on  $\mathfrak{g} = \mathfrak{so}(2n)$  generates the linear transformation group  $H_\parallel^* \subset H^* = \text{Ad}(H)$  that preserves  $\mathfrak{e}$  in  $\mathfrak{m} = \mathfrak{so}(2n)/\mathfrak{u}(n)$ , where  $\mathfrak{e}$  is the element (216) in the Cartan subspace  $\mathfrak{a} \subset \mathfrak{m} = \mathfrak{so}(2n)/\mathfrak{u}(n)$ . This group  $H_\parallel^*$  is given by the matrix representation

$$\begin{pmatrix} C & D \\ -D & C \end{pmatrix} \in SU(2) \times U(n-2) \simeq H_\parallel^* \tag{240}$$

with

$$C = \begin{pmatrix} \cos \lambda & c\lambda^{-1} \sin \lambda & 0 \\ -c\lambda^{-1} \sin \lambda & \cos \lambda & 0 \\ 0 & 0 & \mathbf{C} \end{pmatrix}, \quad D = \begin{pmatrix} d_2 \lambda^{-1} \sin \lambda & d_1 \lambda^{-1} \sin \lambda & 0 \\ d_1 \lambda^{-1} \sin \lambda & -d_2 \lambda^{-1} \sin \lambda & 0 \\ 0 & 0 & \mathbf{D} \end{pmatrix} \tag{241}$$

where

$$\mathbf{C}^t \mathbf{C} + \mathbf{D}^t \mathbf{D} = I_{n-2}, \quad \mathbf{C}^t \mathbf{D} - \mathbf{D}^t \mathbf{C} = 0, \tag{242}$$

$$\lambda^2 = c^2 + d_1^2 + d_2^2. \tag{243}$$

In particular, the subgroup  $U(n-2) \subset H_\parallel^*$  acts on  $\mathfrak{m}_\perp$  by right multiplication

$$\begin{aligned}
&\text{Ad}(\mathbf{C}, \mathbf{D})(\mathbf{b}_\perp, (\mathbf{a}_{1\perp}, \mathbf{b}_{1\perp}, \mathbf{a}_{2\perp}, \mathbf{b}_{2\perp})) \\
&= (\mathbf{b}_\perp, (\mathbf{a}_{1\perp} \mathbf{C}^t + \mathbf{b}_{1\perp} \mathbf{D}^t, \mathbf{b}_{1\perp} \mathbf{C}^t - \mathbf{a}_{1\perp} \mathbf{D}^t, \mathbf{a}_{2\perp} \mathbf{C}^t + \mathbf{b}_{2\perp} \mathbf{D}^t, \mathbf{b}_{2\perp} \mathbf{C}^t - \mathbf{a}_{2\perp} \mathbf{D}^t)) \in \mathfrak{m}_\perp
\end{aligned} \tag{244}$$

where  $(\mathbf{C}, \mathbf{D}) \in U(n-2)$  is defined to be the matrix (240)–(241) with  $c = d_1 = d_2 = 0$  (hence  $\lambda = 0$ ), while the action of the subgroup  $SU(2) \subset H_\parallel^*$  on  $\mathfrak{m}_\perp$  is given by

$$\begin{aligned}
&\text{Ad}(c, d_1, d_2)(\mathbf{b}_\perp, (\mathbf{a}_{1\perp}, \mathbf{b}_{1\perp}, \mathbf{a}_{2\perp}, \mathbf{b}_{2\perp})) \\
&= (\mathbf{b}_\perp, (\cos(\lambda) \mathbf{a}_{1\perp} + \lambda^{-1} \sin(\lambda)(d_2 \mathbf{b}_{1\perp} + c \mathbf{a}_{2\perp} + d_1 \mathbf{b}_{2\perp}), \\
&\quad \cos(\lambda) \mathbf{b}_{1\perp} + \lambda^{-1} \sin(\lambda)(-d_2 \mathbf{a}_{1\perp} - d_1 \mathbf{a}_{2\perp} + c \mathbf{b}_{2\perp}), \\
&\quad \cos(\lambda) \mathbf{a}_{2\perp} + \lambda^{-1} \sin(\lambda)(-c \mathbf{a}_{1\perp} + d_1 \mathbf{b}_{1\perp} - d_2 \mathbf{b}_{2\perp}), \\
&\quad \cos(\lambda) \mathbf{b}_{2\perp} + \lambda^{-1} \sin(\lambda)(-d_1 \mathbf{a}_{1\perp} - c \mathbf{b}_{1\perp} + d_2 \mathbf{a}_{2\perp})) \in \mathfrak{m}_\perp
\end{aligned} \tag{245}$$

where  $(c, d_1, d_2) \in SU(2)$  is defined to be the matrix (240)–(241) with  $\mathbf{C} = I_{n-2}$  and  $\mathbf{D} = 0$ . The action of the subgroups  $U(n-2) \subset H_{\parallel}^*$  and  $SU(2) \subset H_{\parallel}^*$  on  $\mathfrak{m}_{\parallel}$  is respectively given by

$$\begin{aligned} \text{Ad}(\mathbf{C}, \mathbf{D})(\mathbf{a}_{\parallel}, (\mathbf{A}_{\parallel}, \mathbf{B}_{\parallel})) \\ = (\mathbf{a}_{\parallel}, (\mathbf{C}\mathbf{A}_{\parallel}\mathbf{C}^t + \mathbf{D}\mathbf{B}_{\parallel}\mathbf{C}^t + \mathbf{C}\mathbf{B}_{\parallel}\mathbf{D}^t - \mathbf{D}\mathbf{A}_{\parallel}\mathbf{D}^t, \\ - \mathbf{C}\mathbf{A}_{\parallel}\mathbf{D}^t - \mathbf{D}\mathbf{B}_{\parallel}\mathbf{D}^t + \mathbf{C}\mathbf{B}_{\parallel}\mathbf{C}^t - \mathbf{D}\mathbf{A}_{\parallel}\mathbf{C}^t)) \in \mathfrak{m}_{\parallel} \end{aligned} \quad (246)$$

and

$$\text{Ad}(c, d_1, d_2)(\mathbf{a}_{\parallel}, (\mathbf{A}_{\parallel}, \mathbf{B}_{\parallel})) = (\mathbf{a}_{\parallel}, (\mathbf{A}_{\parallel}, \mathbf{B}_{\parallel})) \in \mathfrak{m}_{\parallel}. \quad (247)$$

Composition of these two subgroups yields the group  $H_{\parallel}^* = \text{Ad}(SU(2) \times U(n-2)) \subset \text{Ad}(U(n))$ .

**Proposition A.2.** *The vector space  $\mathfrak{m}_{\perp} \simeq \mathbb{R} \oplus \mathbb{R}^{n-2} \oplus \mathbb{R}^{n-2} \oplus \mathbb{R}^{n-2} \oplus \mathbb{R}^{n-2}$  is a reducible representation of the group  $H_{\parallel}^*$  such that the linear map  $\text{ad}(\mathbf{e})^2$  is given by*

$$\text{ad}(\mathbf{e})^2(\mathbf{b}_{\perp}, (\mathbf{a}_{1\perp}, \mathbf{b}_{1\perp}, \mathbf{a}_{2\perp}, \mathbf{b}_{2\perp})) = -(1/\chi)(4\mathbf{b}_{\perp}, (\mathbf{a}_{1\perp}, \mathbf{b}_{1\perp}, \mathbf{a}_{2\perp}, \mathbf{b}_{2\perp})). \quad (248)$$

The irreducible subspaces in this representation consist of  $(\mathbf{b}_{\perp}, (0, 0, 0, 0)) \simeq \mathbb{R}$  and  $(0, (\mathbf{a}_{1\perp}, \mathbf{b}_{1\perp}, \mathbf{a}_{2\perp}, \mathbf{b}_{2\perp})) \simeq \mathbb{R}^{n-2} \oplus \mathbb{R}^{n-2} \oplus \mathbb{R}^{n-2} \oplus \mathbb{R}^{n-2}$  on which  $\text{ad}(\mathbf{e})^2$  has respective eigenvalues  $-4/\chi$  and  $-1/\chi$ .

**A.2. Vector and matrix identities.** Here we list some basic vector/matrices identities:

$$\mathbf{a} \cdot (\mathbf{b}] \mathbf{E}) = \mathbf{b} \cdot (\mathbf{E}] \mathbf{a}) \quad (249)$$

$$\mathbf{C}] \mathbf{a} = -\mathbf{a}] \mathbf{C} \quad (250)$$

$$\mathbf{a}] (\mathbf{C}] \mathbf{D}) = (\mathbf{a}] \mathbf{C})] \mathbf{D} = \mathbf{D}] (\mathbf{C}] \mathbf{a}) \quad (251)$$

$$\mathbf{a} \cdot (\mathbf{b}] (\mathbf{C}] \mathbf{D})) = \frac{1}{2} \mathbf{D} \cdot ((\mathbf{b}] \mathbf{C}) \wedge \mathbf{a}) = (\mathbf{b}] \mathbf{C}) \cdot (\mathbf{D}] \mathbf{a}) \quad (252)$$

$$(\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{C} = 2\mathbf{a} \cdot (\mathbf{C}] \mathbf{b}) = 2\mathbf{b} \cdot (\mathbf{a}] \mathbf{C}) = 2(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{C} \quad (253)$$

$$\mathbf{C} \cdot \mathbf{D} = \mathbf{C}] \mathbf{D} = \mathbf{D}] \mathbf{C} \quad (254)$$

where  $\mathbf{a}, \mathbf{b}$  are vectors,  $\mathbf{C}, \mathbf{D}$  are antisymmetric matrices, and  $\mathbf{E}$  is any matrix.

#### ACKNOWLEDGEMENTS

S.C.A. is supported by an NSERC research grant.

#### REFERENCES

- [1] S.C. Anco, Preprint (2014).
- [2] A.G.M. Ahmed, S.C. Anco, and E. Asadi, Preprint (2014).
- [3] H. Hasimoto, J. Fluid Mech. 51, 477–485 (1972).
- [4] A. Doliwa, P.M. Santini, Phys. Lett. A 185, 373–384 (1994).
- [5] G. Mari Beffa, J. Sanders, J.-P. Wang, J. Nonlinear Sci. 12, 143–167 (2002).
- [6] S.C. Anco and R. Myrzakulov, J. Geom. Phys. 60 (2010), 1576–1603.
- [7] R. Bishop, Amer. Math. Monthly 82, 246–251 (1975).
- [8] S.C. Anco, J. Geom. Phys. 58 (2008), 1–37.
- [9] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry* Vol. I and II, (Wiley, 1969).
- [10] R.W. Sharpe, *Differential geometry*, Vol. 166 Graduate Texts in Mathematics (Springer-Verlag, 1997).
- [11] H. Guggenheimer, *Differential Geometry* (McGraw Hill, 1963).
- [12] P.J. Olver, *Applications of Lie groups to differential equations*, (Springer-Verlag, 1986).
- [13] S.C. Anco, J. Phys. A: Math. and Gen. 36 (2003), 8623–8638.

- [14] I. Dorfman, *Dirac Structures and Integrability of Nonlinear Evolution Equations* (Wiley, 1993).
- [15] A.P. Fordy and P.P. Kullish, Commun. Math. Phys. 89 (1983) 427–443.
- [16] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces* (Amer. Math. Soc., Providence, 2001).
- [17] J. Lepowsky and G.W. McCollum, Trans. Amer. Math. Soc. 216 (1976), 217-228.